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Distinguishing numbers of Cartesian products of multiple complete graphs

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Abstract

We examine the distinguishing number of the Cartesian product of an arbitrary number of complete graphs. We show that for $u_1 \leq \cdots \leq u_d$ the distinguishing number of the Cartesian product of complete graphs of these sizes is either $\lceil u_d^{1/s} \rceil$ or $\lceil u_d^{1/s} \rceil + 1$ where $s = \Pi_{i=1}^{d-1} u_i$. In most cases, which of these values it is can be explicitly determined.

Keywords: Cartesian product, complete graph, distinguishing number.

Math. Subj. Class.: 05C78, 20B25, 68R10

1 Introduction

The distinguishing number of a graph is the minimum number of labels needed so that the only label preserving automorphism is the identity automorphism. The distinguishing number was introduced by Albertson and Collins in [2]. Albertson [1] observed that the distinguishing number of powers (with respect to Cartesian product) of complete graphs provides an upper bound on the distinguishing number of powers of prime graphs. (Formal definitions will be given in the next section.)

Building on results [3] that the hypercube has distinguishing number 2, Albertson [1] showed that that distinguishing number of the d^{th} power of a complete graph is 2 when $d \geq 4$. This result was extended in [10] and [9] showing that Cartesian powers of complete graphs have distinguishing number 2 except for squares and cubes of K_2 and the square of K_3 all of which have distinguishing number 3. We will look more generally at the distinguishing number of Cartesian products of complete graphs of different sizes.

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For products of graphs of different sizes, the distinguishing number of the Cartesian product of complete graphs on u_1 and u_2 vertices for $u_1 \leq u_2$ was examined in [5] and [7]. It is either $\lceil u_2^{1/u_1} \rceil$ or $\lceil u_2^{1/u_1} \rceil + 1$. In 'most' cases which value it is can be determined explicitly and in a few cases it can be determined using a recursion.

More generally one can talk about the distinguishing number of a group acting on a set as the minimum number of labels needed so that the only label preserving group element is the identity. When the group is the group of automorphisms of a graph and the set is the vertex set this is the distinguishing number of the graph. Chan [4] examined $S_m \times S_n$ acting on $[m] \times [n]$. When $m \neq n$ this is the distinguishing number of the Cartesian product of complete graphs of size m and n hence the results in the previous paragraph apply.

In this paper, we examine the distinguishing number of the Cartesian product of complete graphs of sizes $u_1 \leq u_2 \leq \cdots \leq u_d$. As a tool we will use the closely related distinguishing number of the group $S_{u_1} \times S_{u_2} \times \cdots \times S_{u_d}$ acting on $[u_1] \times [u_2] \times \cdots \times [u_d]$. These are identical problems when the u_i are distinct. We will show the following. Except for $d = 2$ with $u_1 = u_2 = 2$ or $u_1 = u_2 = 3$ and $d = 3$ with $u_1 = u_2 = u_3 = 2$ these two distinguishing numbers are the same. This distinguishing number is either $\lceil u_d^{1/s} \rceil$ or $\lceil u_d^{1/s} \rceil + 1$ where $s = \Pi_{i=1}^{d-1} u_i$. As with the case $d = 2$, in 'most' cases which value it is can be determined explicitly and in a few cases it is can be determined using a recursion. Our proof uses induction via two basic reduction lemmas. The switching lemma extends a version for the Cartesian product of 2 complete graphs and reduces the sizes of the u_i. The collapsing lemma reduces the 'dimension', the number d of factors.

A relatively simple argument for the lower bound, $\lceil u_d^{1/s} \rceil$, is the following. For a c-coloring f of $G \Box K_u$ let f_v be a coloring of G given by $f_v = f(x, v)$. If $u > c^s$ there must be two vertices v, v' in K_n with $f_v = f_{v'}$. Then the automorphism that is constant on G coordinates and transposing v and v' is label preserving. Hence, the distinguishing number is at least $u^{1/s}$ and, since it is integral, at least $\lceil u^{1/s} \rceil$. Then apply this with $G = K_{u_1} \Box \cdots \Box K_{u_{d-1}}$. This approach is found in our Lemma 4.3 as well as earlier papers on distinguishing numbers of Cartesian products. The nice description above is suggested by an anonymous referee.

The upper bound $\lceil u^{1/s} \rceil + 1$ for Cartesian powers of general graphs follows from the upper bound observation noted above on powers of complete graphs (see also Observation 2.5 below and the comments surrounding it) and our upper bound in Corollary 5.1. When $s < u_d$ we do not need our Corollary 5.1 as it follows from the upper bound on Cartesian products of two complete graphs upper bound found independently in [5] and [7].

Hence, the main work in our proof covers the cases $u_d \leq s$. As $\lceil u_d^{1/s} \rceil \leq 2$ for $0 \leq u_d \leq s$, we are establishing an upper bound of 3 on the distinguishing number in these cases. We work with general c rather than $c = 2$ needed for $u_d \leq s$ since, in addition, our proof establishes exactly which of the two bounds holds for 'most' cases of Cartesian products of complete graphs. It is also interesting to note that the determination of the exact value of the distinguishing number depends only on u_d, u_{d-1} and the product $\Pi_{j=1}^{d-2} u_j$.

2 Definitions and background

We begin with basic definitions.

Definition 2.1 (Distinguishing Labeling)**.** Given a set V and a group H acting on V, a labeling $\ell : X \to \{1, 2, \ldots, c\}$ is c-distinguishing if only the identity preserves the labels.

When the set V is the vertex set of a graph and the group is the automorphisms of the graph we will call this a distinguishing labeling of the graph.

Definition 2.2 (Distinguishing Number)**.** Given a set V and a group H acting on V, the distinguishing number $D_H(V)$ is the smallest c such that there is a c-distinguishing labeling. For graph automorphisms, $\mathrm{Aut}(G)$ acting on a graph G with vertex set V we will write $D(G)$ instead of $D_{\mathrm{Aut}(G)}(V)$ and refer to the distinguishing number of the graph.

Definition 2.3. The Cartesian product of graphs G_1 and G_2 on vertex sets V_1 and V_2 respectively, denoted $G_1 \Box G_2$, is the graph with vertex set $V_1 \times V_2$ and an edge between (u_1, u_2) and (v_1, v_2) if and only if either $u_1 = v_1$ and u_2v_2 is an edge of G_2 or $u_2 = v_2$ and u_1v_1 is an edge of G_1. Multiple powers are defined inductively $G_1 \Box G_2 \Box \cdots \Box G_d = (G_1 \Box \cdots \Box G_{d-1}) \Box G_d$.

The d^{th} Cartesian power of a graph is the product of d copies of the graph. That is, $G^d = G \Box G^{d-1}$ with $G^2 = G \Box G$. A graph G is prime with respect to Cartesian product if whenever $G = G_1 \Box G_2$, then either G_1 or G_2 is the trivial graph with a single vertex.

The operation $\Box$ is associative and commutative. Graphs have unique prime factorizations $G = G_1 \Box \cdots \Box G_d$ where each G_i is prime. Graphs G_1 and G_2 are said to be relatively prime if they do not share a common factor.

In order to simplify notation, we will use the following terminology to distinguish the two closely related distinguishing labeling problems that we will examine.

For a Cartesian product $K_{u_1} \Box K_{u_2} \Box \cdots \Box K_{u_d}$ of complete graphs, we can identify the vertices with d-tuples from $[u_1] \times [u_2] \times \cdots \times [u_d]$ with two vertices adjacent if and only if they agree on exactly $d - 1$ coordinates. We can think of a c-labeling as an array of size $u_1, u_2, \ldots, u_d$ with entries from $[c]$. By the Imrich and Miller result noted below, automorphisms correspond to permutations of entries within each dimension along with 'transposes' of same size dimensions. For simplicity in notation, we will refer to this set and group of automorphisms as an array of size $u_1, u_2, \ldots, u_d$ and write $D(Array(u_1, u_2, \ldots, u_d))$ for the distinguishing number.

For distinguishing labelings of $S_{u_1} \times S_{u_2} \times \cdots \times S_{u_d}$ acting on $[u_1] \times [u_2] \times \cdots \times [u_d]$, we can think of the elements as d-tuples from $[u_1] \times [u_2] \times \cdots \times [u_d]$. The group actions correspond to permutations of entries within each dimension. Unlike the previous paragraph, 'transposes' of same size dimensions do not occur. For simplicity in notation we will refer to this set and group of automorphisms as a grid of size $u_1, u_2, \ldots, u_d$ and write $D(Grid(u_1, u_2, \ldots, u_d))$ for the distinguishing number.

For completeness, we make the following trivial observations which follow from the fact that the group acting on the grids is a subgroup of that acting on the arrays.

Observation 2.4. If the array of size $u_1, u_2, \ldots, u_d$ has a distinguishing c-coloring, then the grid of size $u_1, u_2, \ldots, u_d$ has a distinguishing c-coloring. The distinguishing number of the grid of size $u_1, u_2, \ldots, u_d$ is at most the distinguishing number of the array of size $u_1, u_2, \ldots, u_d$. That is, $D(Grid(u_1, u_2, \ldots, u_d)) \leq D(Array(u_1, u_2, \ldots, u_d))$.

As noted above, the distinguishing numbers of arrays and grids of the same size will be shown to be equal except in the following three cases: $d = 2$ with $u_1 = u_2 = 2$ or with $u_1 = u_2 = 3$ and $d = 3$ with $u_1 = u_2 = u_3 = 2$. In each of these small cases, it is straightforward to check that the grid distinguishing number is 2 and the array distinguishing number is 3. For arrays, the first two are noted in [3] and the third in [9].

For grids, the first two are noted in [4] and for the third, observe that the $[2] \times [2] \times [2]$ grid with with entries $a_{i,j,1}$ given by $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and entries $a_{i,j,2}$ given by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is 2-distinguishing.

A result shown independently by Imrich and Miller (see [8]) is that if G is connected and $G = G_1 \Box \cdots \Box G_d$ is its prime decomposition, then every automorphism of G is generated by the automorphisms of the factors and the transpositions of isomorphic factors. Using this we make the following observation which extends those of Albertson [1] and Imrich, Jerebic, and Klavžar [7]. As with the Observation 2.4, it follows as the group acting on the vertex set of the graph is a subgroup of that acting on the array.

Observation 2.5. If $G = G_1 \Box G_2 \Box \cdots \Box G_d$ with the G_i relatively prime and $|V(G_i)| = u_i$, then $D(G_1 \Box G_2 \Box \cdots \Box G_d) \leq D(Array(u_1, u_2, \ldots, u_d))$.

As a consequence of our results, we will see that $D(Array(u_1, \ldots, u_{i-2}, u_{i-1}, u_i, u_{i+1}, \ldots, u_s)) \leq D(Array(u_1, \ldots, u_{i-2}, (u_{i-1}u_i), u_{i+1}, \ldots, u_s))$. Thus, if the sizes of the factors in the prime factorization of $G_1 \Box G_2 \Box \cdots \Box G_d$ are $u_1, u_2, \ldots, u_{d'}$, then $D(G_1 \Box G_2 \Box \cdots \Box G_d) \leq D(Array(u_1, u_2, \ldots, u_{d'}))$. Hence, we can drop the relatively prime assumption in Observation 2.5.

We will also use the following definitions and notation. Examples of switching and collapsing in 2 dimensions are described in the next section.

For automorphisms of grids we will write $(\pi_1, \pi_2, \ldots, \pi_d)$ where π_i is a permutation of $[u_i]$. For arrays we will write automorphisms as $(\sigma; \pi_1, \ldots, \pi_d)$ where the π_i are as for grids and σ is a permutation of $[d]$ (such that only positions with equal sizes can be permuted). We will apply the π_i first and then permute the dimensions.

Definition 2.6 (Switching)**.** For a c-labeling A with entries $a(i_1, i_2, \ldots, i_d)$ of an array (grid) of size $u_1, u_2, \ldots, u_d$, the dimension i subarrays are the $(d-1)$ dimensional arrays A_k of size $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_d)$ given by $a_k(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_d) = a(j_1, \ldots, j_{i-1}, k, j_{i+1}, \ldots, j_d)$ for $k = 1, 2, \ldots, u_i$.

If A is a c-labeling of an array of size $u_1, u_2, \ldots, u_d$ and the dimension i subarrays are distinct, then the dimension i complement A_i^* is the size $u_1, \ldots, u_{i-1}, c^p - u_i, u_{i+1}, \ldots, u_d$ array where $p = u_1 \cdots u_{i-1} u_{i+1} \cdots u_d$, whose set of dimension i subarrays is the set of all c labelings of arrays of size $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t)$ that do not appear as dimension i subarrays of A.

For example, in two dimensions the dimension 1 subarrays are the rows and the dimension 2 subarrays are the columns. Also, in two dimensions, the dimension 2 complement A_2^* is the size $(u_1, c^{u_1} - u_2)$ array consisting of all 'possible' columns (size u_1 strings with entries from $[c]$) that do not appear in A.

Technically, the complement would depend on the ordering of the complementary subarrays but as this will not affect what we do we will assume some fixed ordering and refer to 'the' complement.

Definition 2.7 (Collapsing)**.** For a c-labeling A with entries $a(i_1, i_2, \ldots, i_d)$ of an array of size $u_1, u_2, \ldots, u_d$ and $S = \{k+1, \ldots, d\}$ the array A^S obtained by collapsing the dimensions in S has $d - |S| + 1$ dimensions with sizes $(u_1, u_2, \ldots, u_k, q)$ where $q = u_{k+1} \cdots u_d$. It is defined as follows: Let g be any bijection from $[u_{k+1}] \times \cdots \times [u_d]$ to $[u_{k+1} \cdots u_d]$. The entries of A^S are then $a^S(j_1 \ldots, j_k, g) = a(j_1, \ldots, j_k, j_{k+1}, \ldots, j_d)$ where $g = g(j_{k+1}, \ldots, j_d)$.

For any subset $S \subset [d]$ of indices, the array A_S obtained by collapsing the dimensions in S is defined in a similar manner and has $d - |S| + 1$ dimensions with sizes u_i for $i \notin S$ and one dimension of size $\Pi_{i \in S} u_i$.

Technically, the collapse would depend on the choice of g but as this will not affect what we do we will assume some fixed g and refer to 'the' collapse.

3 Reduction lemmas

We will make use of two basic lemmas to relate distinguishing labelings of one size to another. This will allow for induction to determine values of c for which arrays and grids of given sizes have distinguishing c-labelings.

Consider a c-labeling A of the size u_1, u_2 grid with distinct columns. The dimension 2 complement A_2^* has size $u_1, c^{u_1} - u_2$ and consists of all $c^{u_1} - u_2$ possible columns that do not appear in A. An automorphism (π_1, π_2) of A that preserves labels maps the columns of A to columns of A under π_1. Hence, applying π_1 to A_2^* must map columns of A_2^* to columns of A_2^*. So we can find a permutation π_2' of $[c^{u_1} - u_2]$ so that (π_1, π_2') is a automorphism of A_2^* that preserves labels. So, we observe that there is a distinguishing labeling of the size u_1, u_2 grid if and only if there is a distinguishing labeling of the size $u_1, c^{u_1} - u_2$ grid. For arrays, we get the same result except that when $u_1 = u_2$ there is the possibility of an automorphism that transposes rows and columns of A that does not correspond to an automorphism of A_2^*. So, in this case, we get only that if there is a distinguishing labeling of the size $u_1, c^{u_1} - u_2$ array and $u_1 \neq u_2$, then there is a distinguishing labeling of the size u_1, u_2 array. This idea will be presented as the switching lemma below.

The idea in the switching lemma (in two dimensions) below has appeared in a number of earlier papers on distinguishing number. It appears in some form (at least) in [4], [5], [7] [9] and [6]. We use the name *switching lemma* given in [7].

For simplicity in notation, we state the following lemma for switching in dimension d. By permuting the labels on the dimensions, a similar statement holds for switching on any dimension. We state versions for both grids and arrays. The proofs are essentially the same as in the two dimensional case. For completeness, we give a proof for the grid version.

Lemma 3.1 (Grid Switching Lemma)**.** Let $u_1 \leq u_2 \leq \cdots \leq u_d$ with $u_d < c^p$ where $p = u_1 u_2 \cdots u_{d-1}$. There is a c-distinguishing labeling of a grid of size $u_1, u_2, \ldots, u_d$ if and only if there is a c-distinguishing labeling of a grid of size $u_1, u_2, \ldots, u_{d-1}, c^p - u_d$.

Proof. Consider a c-labeling A of the size $u_1, u_2, \ldots, u_d$ grid with distinct $(d-1)$-dimensional subgrids. The dimension d complement A_d^* has size $u_1, u_2, \ldots, c^p - u_d$ and consists of all $c^p - u_d$ possible $(d-1)$-dimensional subgrids that do not appear in A. An automorphism $(\pi_1, \pi_2, \ldots, \pi_m)$ of A that preserves labels maps the $(d-1)$-dimensional subgrids of A to the $(d-1)$-dimensional subgrids of A under $\pi_{sub} = (\pi_1, \pi_2, \ldots, \pi_{d-1})$. Hence, applying π_{sub} to A_d^* must map the $(d-1)$-dimensional subgrids of A_d^* to the $(d-1)$-dimensional subgrids of A_d^*. So, we can find a permutation π_d' of $[c^p - u_d]$ so that $(\pi_1, \pi_2, \ldots, \pi_{d-1}, \pi_d')$ is an automorphism of A_d^* that preserves labels. □

We state the following without proof as it is identical to the grid switching lemma.

Lemma 3.2 (Array Switching Lemma)**.** If there is a c-distinguishing labeling of an array of size $u_1, u_2, \ldots, u_{m-1}, c^p - u_m$ where $p = u_1 u_2 \cdots u_{m-1}$ and $u_m \neq u_k$ for all $k = 1, 2, \ldots, m-1$, then there is a c-distinguishing labeling of an array of size $u_1, u_2, \ldots, u_m$.

Our second reduction lemma, which we will call the *collapsing lemma*, allows a reduction in the number of dimensions. It is based on the following simple observation. Consider a c-labeling A of the size u_1, u_2 array. We can write the rows one after another in a string of length $u_1 u_2$ (i.e., 1-dimensional array $A_{\{1,2\}}$ of size $u_1 u_2$). An automorphism (π_1, π_2) of A permutes entries of A and hence there is a corresponding permutation $\pi_{1,2}$ of the entries of $A_{\{1,2\}}$. Thus, if $A_{\{1,2\}}$ is a distinguishing c-labeling then so is A. Note that the converse does not hold (indeed in two dimensions the entries need to be distinct for $A_{\{1,2\}}$ to be distinguishing).

For higher dimensions, we need to be a little careful with equal sizes. For example, consider a c-labeling A of size (u_1, u_2, u_3) with $u_1 = u_2$. An automorphism of A that transposes dimensions 1 and 2 does not necessarily correspond to an automorphism of $A_{\{2,3\}}$, obtained by collapsing dimensions 2 and 3. So, if $u_i = u_j$, then we must either include both in the dimension we collapse or exclude both.

As with the switching lemma, for simplicity in notation we state the collapsing lemma for collapsing dimensions $k+1, \ldots, d$. By permuting labels of the dimensions, a similar statement holds for collapsing any set S as long as $u_i = u_j$ implies either $i, j \in S$ or $i, j \notin S$.

Lemma 3.3 (Collapsing Lemma)**.** If there is a distinguishing c-labeling of the $(k+1)$ dimensional array of size $(u_1, u_2, \ldots, u_k, (u_{k+1} \cdots u_d))$ and for $i \leq k$ and $j > k$ we have $u_i \neq u_j$, then there is a distinguishing c-labeling of the d dimensional array of size $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_d)$.

Proof. We prove the contrapositive. Let A be a c-labeling of the d-dimensional array of size $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_d)$ and define A^S as in definition 2.7 with $S = \{k+1, \ldots, d\}$.

Let A^S be a distinguishing c-labeling of $(u_1, u_2, \ldots, u_k, (u_{k+1} \cdots u_d))$ and let A be the associated c-labeling of $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_d)$. An automorphism $(\sigma; \pi_1, \pi_2, \ldots, \pi_d)$ of A can be used to define an automorphism $(\sigma'; \pi_1, \ldots, \pi_k, \pi_{k+1})$ of A^S in the 'obvious' way. We have σ' the same as σ on $\{1, 2, \ldots, k\}$ with $k+1$ fixed. The automorphism produces some permutation of the entries in dimensions $k+1, \ldots, d$ which we call π'_{k+1}. It is straightforward to see that we get an automorphism of A^S. Hence, if A is not a distinguishing c-labeling of $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_t)$, then A^S is not a distinguishing c-labeling of $(u_1, u_2, \ldots, u_k, (u_{k+1} \cdots u_t))$. □

We observe that there is a corresponding collapsing lemma for grids without the size conditions. We do not need this in what follows so we will not formally state it.

4 Distinguishing labelings

In this section we state a theorem describing those values of c for which there are distinguishing labelings of arrays and grids of a given size. We first give lemmas providing upper and lower bounds.

The following lemma will be used several times in the proof of Lemma 4.2.

Lemma 4.1. Let d be an integer greater than 3. If the array $(u_1, u_2, \ldots, u_{d-1})$ has a distinguishing 2-labeling and if $u_d \leq u_1 u_2 \cdots u_{d-1} + 1$, then $(u_1, u_2, \ldots, u_{d-1}, u_d)$ also has a distinguishing 2-labeling.

Proof. Suppose that $(u_1, u_2, \ldots, u_{d-1})$ has a distinguishing 2-labeling; call it ℓ. We define the weight of a generic 2-labeling to be the total number of times the label one is used in the labeling. Because there are $u_1 u_2 \cdots u_{d-1} + 1$ 2-labelings of $(u_1, u_2, \ldots, u_{d-1})$ with different weights, we can construct a distinguishing 2-labeling of $(u_1, u_2, \ldots, u_{d-1}, u_d)$ by taking ℓ together with any $u_d - 1$ of the other remaining $u_1 u_2 \cdots u_{d-1}$ 2-labelings of $(u_1, u_2, \ldots, u_{d-1})$ with different weights. □

Lemma 4.2. For integers $c \geq 2$, $d \geq 3$ and $1 \leq u_1 \leq u_2 \leq \cdots \leq u_d$ with $u_d \leq c^{u_1 u_2 \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil - 1$ there is a distinguishing c-labeling of the $(u_1, u_2, \ldots, u_d)$ array.

Proof. The proof will proceed by induction on the number d of dimensions and by induction on u_d. We will use the case $d = 2$ as the basis. For $d = 2$ and $u_1 \leq u_2$, if $u_2 \leq c^{u_1} - \lceil \log_c u_1 \rceil - 1$, then the (u_1, u_2) array has a distinguishing c-labeling ([5], [7]). This corresponds to the formula above if we interpret the product $u_1 \cdots u_{d-2}$ as an empty product equal to 1 when $d = 2$.

We will establish the result for $d = 3$ and $d = 4$ separately and then proceed to the case $d \geq 5$. For each d, several cases are needed to determine where collapsing can occur.

$d = 3$: We first establish base cases for induction on u_3. Note that the collapsing lemma implies that $(2, 2, 3)$ has a distinguishing 2-labeling (collapsing the first two terms we have a size $(4, 3)$ array which has a distinguishing 2-labeling). It is easy to verify that the two 2-labelings of $(3, 3)$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

provide us with an explicit distinguishing 2-labeling of $(2, 3, 3)$. The first array giving the $a_{1,j,k}$ entries and the second the $a_{2,j,k}$ entries. Finally, a result of Klavžar and Zhu [10] tells us that $(3, 3, 3)$ has a distinguishing 2-labeling.

Case 1: $(u_1 < u_2 < u_3)$

(i) If $u_1 u_2 < u_3$, then $((u_1 u_2), u_3)$ has a distinguishing c-labeling if $u_3 \leq c^{u_1 u_2} - \lceil \log_c u_1 u_2 \rceil - 1$. By the collapsing lemma, (u_1, u_2, u_3) also has a distinguishing c-labeling.

If $c^{u_1 u_2} - \lceil \log_c u_1 u_2 \rceil - 1 < u_3 \leq c^{u_1 u_2} - \lceil \frac{\log_c u_2}{u_1} \rceil - 1$, then switch in dimension 3 and consider (u_1, u_2, z) where $\lceil \frac{\log_c u_2}{u_1} \rceil + 1 \leq z \leq \lceil \log_c u_1 u_2 \rceil < u_3$.

(a) If $2 \leq u_1 < u_2 \leq z$, then, as $z < u_3 \leq c^{u_1 u_2} - \lceil \frac{\log_c u_2}{u_1} \rceil - 1$, (u_1, u_2, z) has a distinguishing c-labeling by induction on u_3. Thus, by the switching lemma, (u_1, u_2, u_3) also has a distinguishing c-labeling.

(b) If $u_1 < z < u_2$, then it is always true that $u_2 \leq c^{z u_1} - \lceil \frac{\log_c z}{u_1} \rceil - 1$. [$z \geq \frac{\log_c u_2}{u_1} + 1$ implies that $c^{z u_1} \geq u_2 c^{u_1}$. Thus, $c^{z u_1} \geq u_2 c^{u_1}$ and $z < u_2$ imply that $u_2 \leq u_2 c^{u_1} - u_2 - 1 \leq c^{z u_1} - \lceil \frac{\log_c z}{u_1} \rceil - 1$.] Hence, by induction on u_3 and the switching lemma, (u_1, u_2, u_3) has a distinguishing c-labeling.

(c) If $z < u_1 < u_2$, then it is always true that $u_2 \leq c^{zu_1} - \lceil \frac{\log_c u_1}{z} \rceil - 1$. [$z \geq \frac{\log_c u_2}{u_1} + 1$ implies that $c^{zu_1} \geq u_2 c^{u_1}$. Thus, $c^{zu_1} \geq u_2 c^{u_1}$ implies that $u_2 \leq u_2 c^{u_1} - u_2 - 1 \leq u_2 c^{u_1} - u_1 - 1 \leq c^{zu_1} - \lceil \frac{\log_c u_1}{z} \rceil - 1$.] Hence, by induction on u_3 and the switching lemma, (u_1, u_2, u_3) has a distinguishing c-labeling.

(d) If $u_1 = z < u_2$ and $u_1^2 < u_2$, then it is always true that $u_2 \leq c^{zu_1} - \lceil \frac{\log_c z}{z} \rceil - 1 = c^{zu_1} - 2$. [$z \geq \frac{\log_c u_2}{u_1} + 1$ implies that $c^{zu_1} \geq u_2 c^{u_1}$. Thus, $c^{zu_1} \geq u_2 c^{u_1}$ implies that $u_2 \leq u_2 c^{u_1} - 2 \leq c^{zu_1} - 2$.] It follows that (u_1^2, u_2) has a distinguishing c-labeling. Thus, by the collapsing lemma, so does (u_1, z, u_2).

(e) If $u_1 = z < u_2$ and $u_1^2 = u_2$, then (u_2, u_2) has a distinguishing 2-labeling via the $d = 2$ result. Thus, by the collapsing lemma, (u_1, z, u_2) also has a distinguishing 2-labeling. Hence, (u_1, z, u_2) also has a distinguishing c-labeling.

(f) Finally, if $u_1 = z < u_2$ and $u_2 < u_1^2$, then it is always true that $u_1^2 \leq 2^{u_2} - \lceil \log_2 u_2 \rceil - 1$. [It is easy to check that this inequality holds for $u_2 = 3, 4$. For $u_2 \geq 5$, a simple inductive argument implies that $u_2^2 \leq 2^{u_2} - u_2 - 1$. The desired inequality now follows.] It follows that (u_2, u_1^2) has a distinguishing 2-labeling. Thus, by the collapsing lemma, so does (u_1, z, u_2). Therefore, (u_1, z, u_2) also has a distinguishing c-labeling.

Thus, in any case, (u_1, u_2, u_3) also has a distinguishing c-labeling by the switching lemma.

(ii) If $u_3 \leq u_1 u_2$, then since $u_1 u_2 \leq 2^{u_1 u_2} - \lceil \frac{\log_2 u_2}{u_1} \rceil - 1$ always holds, (u_1, u_2, u_3) has a distinguishing 2-labeling by induction on dimension and by the collapsing lemma. [$u_1 < u_2$ implies that $u_2 = u_1 + m$ for some $m \geq 1$. Using first semester calculus, one can show that the function $f(x) = 2^{x^2+mx} - 1 - x^2 - (m+1)x - m$ is strictly increasing for $x \geq 2$ (and m fixed). Since $f(2) > 0$, the desired inequality now follows.]

Case 2: $(u_1 < u_2 = u_3)$

First note that it has already been shown that $(2, 3, 3)$ has a distinguishing 2-labeling. Next, recall that it is known (by the $d = 2$ case) that if $4 \leq u_2 = u_3$, then (u_2, u_3) has a distinguishing 2-labeling. Thus, Lemma 4.1 implies that $(u_1 < u_2 = u_3)$ also has a distinguishing 2-labeling.

Case 3: $(u_1 = u_2 = u_3)$

This case follows from results in [9].

Case 4: $(u_1 = u_2 < u_3)$, where $u_2 + 1 \leq u_3 \leq c^{u_2^2} - \lceil \frac{\log_c u_2}{u_1} \rceil - 1$

(i) For $(u_1 = u_2 < u_2 + \varepsilon)$, $1 \leq \varepsilon \leq u_2^2 - u_2$, we collapse and consider $(u_2 + \varepsilon, u_2^2)$. As we will show, $u_2^2 \leq 2^{u_2+\varepsilon} - \lceil \log_2(u_2 + \varepsilon) \rceil - 1$ holds for all $u_2 \geq 2$. Thus, $(u_2 + \varepsilon, u_2^2)$ has a distinguishing 2-labeling by induction on dimension and hence $(u_1 = u_2 < u_2 + \varepsilon)$ has a distinguishing 2-labeling by the collapsing lemma.

[$u_2^2 \leq 2^{u_2+\varepsilon} - \lceil \log_2(u_2 + \varepsilon) \rceil - 1$ can be verified by hand for the cases $u_2 = 2, 3, 4$. For $u_2 \geq 5$, one can prove by induction that $u_2^2 \leq 2^{u_2+1} - u_2^2 - 1$. Our result then follows.]

(ii) For $(u_1 = u_2 < u_2^2 + 1)$ through $(u_1 = u_2 < c^{u_2^2} - \lceil \log_c u_2^2 \rceil - 1)$, we collapse and look at (u_2^2, z), $u_2^2 + 1 \leq z \leq c^{u_2^2} - \lceil \log_c u_2^2 \rceil - 1$. By induction on dimension, (u_2^2, z) has a distinguishing c-labeling.

(iii) Finally, we see by the switching lemma that $(u_1 = u_2 < c^{u_2^2} - \lceil \log_c u_2^2 \rceil)$ through $(u_1 = u_2 < c^{u_2^2} - \lceil \frac{\log_c u_2}{u_2} \rceil - 1)$ all have a distinguishing c-labeling. Note that one case not covered here is $(2, 2, 14)$ ($(2, 2, 2)$ does not have a distinguishing 2-labeling). However, since $(4, 14)$ has a distinguishing 2-labeling, the collapsing lemma shows that $(2, 2, 14)$ does too.

$d = 4$: The case $(2, 2, 2, 2)$ will serve as the basis step for induction on u_4.

Case 1: $(u_1 < u_2 < u_3 < u_4)$

(i) If $u_1 u_2 \leq u_3$, then, by induction on dimension, $(u_1 u_2, u_3, u_4)$ has a distinguishing c-labeling if $u_4 \leq c^{u_1 u_2 u_3} - \lceil \frac{\log_c u_3}{u_1 u_2} \rceil - 1$.

(ii) If $u_3 < u_1 u_2 = u_4$, then we use Lemma 4.1 to build a distinguishing 2-labeling for $(u_3, u_1 u_2, u_4)$ from a distinguishing 2-labeling of $(u_1 u_2, u_4)$.

(iii) If $u_3 < u_1 u_2 < u_4$, then $(u_3, u_1 u_2, u_4)$ has a distinguishing c-labeling if $u_4 \leq c^{u_1 u_2 u_3} - \lceil \frac{\log_c u_1 u_2}{u_3} \rceil - 1$. If $c^{u_1 u_2 u_3} - \lceil \frac{\log_c u_1 u_2}{u_3} \rceil - 1 < u_4 \leq c^{u_1 u_2 u_3} - \lceil \frac{\log_c u_3}{u_1 u_2} \rceil - 1$, then we switch in the last dimension and look at $\lceil \frac{\log_c u_3}{u_1 u_2} \rceil + 1 \leq z < \lceil \frac{\log_c u_1 u_2}{u_3} \rceil + 1$. Note that the last string of inequalities is equivalent to $2 \leq z < 3$. Thus, the only value of z that we need to check is $z = 2$.

To decide whether or not $(2, u_3, u_1 u_2)$ has a distinguishing c-labeling, we switch in the second dimension and look at $(2, c^{2 u_1 u_2} - u_3, u_1 u_2)$. Since $u_3 < u_1 u_2$, we have $u_1 u_2 \leq c^{2 u_1 u_2} - u_3$. Now, since $u_3 \geq \lceil \log_c u_2 \rceil + 1 \geq \lceil \frac{\log_c u_1 u_2}{2} \rceil + 1$, it follows that $c^{2 u_1 u_2} - u_3 \leq c^{2 u_1 u_2} - \lceil \frac{\log_c u_1 u_2}{2} \rceil - 1$. Hence, $(2, u_1 u_2, c^{2 u_1 u_2} - u_3)$ has a distinguishing c-labeling. Therefore, $(2, u_3, u_1 u_2)$ does too.

(iv) If $u_4 < u_1 u_2$, then it is easily seen that $u_1 u_2 \leq 2^{u_3 u_4} - \lceil \frac{\log_2 u_4}{u_3} \rceil - 1$. [$u_1 u_2 < u_3 u_4$ and $u_4 < u_1 u_2$ imply the desired inequality.] Hence, $(u_3, u_4, u_1 u_2)$ has a distinguishing 2-labeling.

Case 2: $(u_1 \leq u_2 < u_3 = u_4)$

(i) If $u_1 u_2 < u_3$, then $u_4 \leq 2^{u_1 u_2 u_3} - \lceil \frac{\log_2 u_3}{u_1 u_2} \rceil - 1$ always holds. [One can show by induction on u_4 that $2 u_4 \leq 2^{u_1 u_2 u_4} - 1$. The desired inequality follows from the last statement.] Hence, by the collapsing lemma, (u_1, u_2, u_3, u_4) has a distinguishing 2-labeling.

(ii) Similarly, if $u_3 \leq u_1 u_2$, then $u_4 = u_3 \leq u_1 u_2 < u_4^2 \leq 2^{u_4^2} - 2$ always holds. Hence, by the collapsing lemma, (u_1, u_2, u_3, u_4) has a distinguishing 2-labeling.

Case 3: $(u_1 < u_2 = u_3 < u_4)$

(i) If $u_2^2 < u_4$, then (u_1, u_2, u_3, u_4) has a distinguishing c-labeling, by the collapsing lemma, if $u_4 \leq c^{u_1 u_2^2} - \lceil \frac{\log_c u_2^2}{u_1} \rceil - 1$. If $c^{u_1 u_2^2} - \lceil \frac{\log_c u_2^2}{u_1} \rceil - 1 < u_4 \leq c^{u_1 u_2^2} - \lceil \frac{\log_c u_3}{u_1 u_2} \rceil - 1$, then we switch in the last dimension and look at $\lceil \frac{\log_c u_3}{u_1 u_2} \rceil + 1 \leq z < \lceil \frac{\log_c u_2^2}{u_1} \rceil + 1$. The last string of inequalities implies that $2 \leq z < 2u_2$.

(a) If $2 \leq z \leq u_2$, then (u_1, z, u_2, u_3) (or (z, u_1, u_2, u_3)) has a distinguishing 2-labeling as long as $u_3 \leq 2^{z u_1 u_2} - \lceil \frac{\log_2 u_2}{z u_1} \rceil - 1$. [$u_3 \leq 2^{z u_1 u_3} - u_3 - 1$ is always true by induction, since $z u_1 \geq 4$ and $u_2 = u_3$. The desired inequality now follows.]

(b) If $u_2 < z < 2u_2$, then (u_1, u_2, u_3, z) has a distinguishing 2-labeling as long as $z < 2u_2 \leq 2^{u_1 u_2^2} - \lceil \frac{\log_2 u_3}{u_1 u_2} \rceil - 1$. [Since $u_2 = u_3$, one can prove via induction that $2u_2 \leq 2^{u_1 u_2^2} - u_2 - 1$ holds for $u_2 \geq 3$. The desired inequality now follows.]

(ii) If $u_4 \leq u_2^2$, then it is always true that $u_4 \leq 2^{u_2^2} - \lceil \frac{\log_2 u_2}{u_2} \rceil - 1 = 2^{u_2^2} - 2$. Hence, (u_2, u_2, u_4) has a distinguishing 2-labeling. Therefore, Lemma 4.1 implies that (u_1, u_2, u_3, u_4) has a distinguishing 2-labeling.

Case 4: $(u_1 = u_2 < u_3 \leq u_4)$

(i) If $u_2^2 \leq u_3$, then $u_4 \leq c^{u_3 u_2^2} - \lceil \frac{\log_c u_3}{u_2^2} \rceil - 1$ implies that (u_2^2, u_3, u_4) has a distinguishing c-labeling.

(ii) If $u_3 < u_2^2 \leq u_4$, then $u_4 \leq c^{u_3 u_2^2} - \lceil \frac{\log_c u_2^2}{u_3} \rceil - 1$ implies that (u_3, u_2^2, u_4) has a distinguishing c-labeling. If $c^{u_3 u_2^2} - \lceil \frac{\log_c u_2^2}{u_3} \rceil - 1 < u_4 \leq c^{u_3 u_2^2} - \lceil \frac{\log_c u_3}{u_2^2} \rceil - 1$, then switch in the last dimension and look at $\lceil \frac{\log_c u_3}{u_2^2} \rceil + 1 \leq z < \lceil \frac{\log_c u_2^2}{u_3} \rceil + 1$. Note that the last string of inequalities implies that $2 \leq z < 3$. Thus the only value of z that we need to consider is $z = 2$.

Hence, we consider $(2, u_1, u_2, u_3)$. By induction on u_4, $(2, u_1, u_2, u_3)$ has a distinguishing c-labeling as $u_3 < u_2^2 \leq c^{2u_2^2} - \lceil \frac{\log_c u_2}{2u_2} \rceil - 1 = c^{2u_2^2} - 2$.

(iii) If $u_3 \leq u_4 < u_2^2$, then (u_3, u_4, u_2^2) has a distinguishing 2-labeling as $u_2^2 \leq 2^{u_3 u_4} - \lceil \frac{\log_2 u_4}{u_3} \rceil - 1$. [The inequality we want follows from the inequality $u_2^2 < 2^{u_2^2} - u_2^2 - 1$.]

Case 5: $(u_1 \leq u_2 = u_3 = u_4)$ and $(u_1 = u_2 = u_3 < u_4)$

(i) For $(u_1 < u_2 = u_3 = u_4)$, we note that (u_2, u_3, u_4) has a distinguishing 2-labeling. Hence, by Lemma 4.1, (u_1, u_2, u_3, u_4) has one too.

(ii) The case $(u_1 = u_2 = u_3 = u_4)$ is already known.

(iii) For $(u_1 = u_2 = u_3 < u_3 + 1)$, we note that $(u_2, u_3, u_3 + 1)$ has a distinguishing 2-labeling. Hence, by Lemma 4.1, $(u_1, u_2, u_3, u_3 + 1)$ has one too.

(iv) For $(u_1 = u_2 = u_3 < u_3 + \varepsilon)$, $2 \leq \varepsilon \leq u_3^3 - u_3$, we collapse and consider $(u_3 + \varepsilon, u_3^3)$. As we will show, $u_3^3 \leq 2^{u_3 + \varepsilon} - \lceil \log_2(u_3 + \varepsilon) \rceil - 1$ holds for all $u_3 \geq 2$, except for $(u_3, \varepsilon) = (4, 2)$ and $(u_3, \varepsilon) = (5, 2)$. By Lemma 4.1, $(4, 4, 4, 6)$ and $(5, 5, 5, 7)$ both have distinguishing 2-labelings. For all other cases, we observe that $(u_3+\varepsilon, u_3^3)$ has a distinguishing 2-labeling by induction on dimension and hence $(u_1 = u_2 = u_3 < u_3 + \varepsilon)$ has a distinguishing 2-labeling by the collapsing lemma.

[$u_3 \le 2^{u_3+\varepsilon} - \lceil \log_2(u_3+\varepsilon)\rceil - 1$ can be verified by hand for the cases $2 \le u_3 \le 9$. For $u_3 \ge 10$, one can prove by induction that $u_3^3 \le 2^{u_3+1} - u_3^3 - 1$. Our result then follows.]

(v) For $(u_1 = u_2 = u_3 < u_3^3 + 1)$ through $(u_1 = u_2 = u_3 < c^{u_3^3} - \lceil \log_c u_3^3 \rceil - 1)$, we look at (u_3^3, z), $u_3^3 + 1 \le z \le c^{u_3^3} - \lceil \log_c u_3^3 \rceil - 1$.

(vi) The switching lemma takes care of $(u_1 = u_2 = u_3 < c^{u_3^3} - \lceil \log_c u_3^3 \rceil)$ through $(u_1 = u_2 = u_3 < c^{u_3^3} - \lceil \frac{\log_c u_3}{u_1 u_2} \rceil - 1)$.

$d \ge 5$: The case $(2, 2, 2, 2, \ldots, 2)$ will serve as the basis step for induction on u_d.

Case 1: $(u_1 \le u_2 < u_3 \le u_4 \le \cdots \le u_{d-1} \le u_d)$

(i) If $u_1u_2 \le u_{d-1}$, then $(u_3, u_4, \ldots, u_1u_2, \ldots, u_{d-1}, u_d)$ has a distinguishing c-labeling if $u_d \le c^{u_1u_2\cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1u_2\cdots u_{d-2}} \rceil - 1$.

(ii) If $u_{d-1} < u_1u_2 \le u_d$, then $(u_3, u_4, \ldots, u_{d-1}, u_1u_2, u_d)$ has a distinguishing c-labeling if $u_d \le c^{u_1u_2\cdots u_{d-1}} - \lceil \frac{\log_c u_1u_2}{u_3u_4\cdots u_{d-1}} \rceil - 1$. Since $\lceil \frac{\log_c u_1u_2}{u_3u_4\cdots u_{d-1}} \rceil = \lceil \frac{\log_c u_{d-1}}{u_1u_2\cdots u_{d-2}} \rceil = 1$, we never have to worry about the case $c^{u_1u_2\cdots u_{d-1}} - \lceil \frac{\log_c u_1u_2}{u_3u_4\cdots u_{d-1}} \rceil - 1 < u_d$.

(iii) If $u_d < u_1u_2$, then $(u_3, u_4, \ldots, u_{d-1}, u_d, u_1u_2)$ has a distinguishing c-labeling as $u_1u_2 \le c^{u_3u_4\cdots u_d} - \lceil \frac{\log_c u_d}{u_3u_4\cdots u_{d-1}} \rceil - 1 = c^{u_3u_4\cdots u_d} - 2$. Because $u_d < u_1u_2$, we do not have to worry about the case $c^{u_3u_4\cdots u_d} - \lceil \frac{\log_c u_d}{u_3u_4\cdots u_{d-1}} \rceil - 1 \le u_d$.

Case 2: $(u_1 = u_2 = \cdots = u_k < u_{k+1} \le \cdots \le u_{d-1} \le u_d)$, where $2 \le k \le d-2$

(i) If $u_1^k \le u_{d-1}$, then $(u_{k+1}, \ldots, u_1^k, \ldots, u_{d-1}, u_d)$ has a distinguishing c-labeling if $u_d \le c^{u_1^k\cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1^k\cdots u_{d-2}} \rceil - 1$.

(ii) If $u_{d-1} < u_1^k \le u_d$, then $(u_{k+1}, \ldots, u_{d-1}, u_1^k, u_d)$ has a distinguishing c-labeling if $u_d \le c^{u_1^k\cdots u_{d-1}} - \lceil \frac{\log_c u_1^k}{u_{k+1}\cdots u_{d-1}} \rceil - 1$. If $c^{u_1^k\cdots u_{d-1}} - \lceil \frac{\log_c u_1^k}{u_{k+1}\cdots u_{d-1}} \rceil - 1 < u_d \le c^{u_1^k\cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1u_2\cdots u_{d-2}} \rceil - 1 = c^{u_1^k\cdots u_{d-1}} - 2$, then switch and look at

$$2 = \lceil \frac{\log_c u_{d-1}}{u_1u_2\cdots u_{d-2}} \rceil + 1 \le z < \lceil \frac{\log_c u_1^k}{u_{k+1}\cdots u_{d-1}} \rceil + 1 = k + 1 < u_1^k.$$

(a) If $z < u_{d-1}$, then look at $(u_1, u_2, \ldots, z, \ldots, u_{d-1})$. Note that $(u_1, u_2, \ldots, z, \ldots, u_{d-1})$ has a distinguishing c-labeling if $u_{d-1} \le c^{u_1^k u_{k+1}\cdots z} - \lceil \frac{\log_c \delta}{\varepsilon} \rceil - 1$, where $\varepsilon = u_1u_2\cdots u_{d-2}$ if $\delta = z$ and $\varepsilon = u_1u_2\cdots u_{d-3}z$ if $\delta = u_{d-2}$. Since, in either case, $\lceil \frac{\log_c \delta}{\varepsilon} \rceil \le u_{d-1}$, it follows that $u_{d-1} \le c^{u_1^k u_{k+1}\cdots z} - \lceil \frac{\log_c \delta}{\varepsilon} \rceil - 1$ always holds.

(b) If $u_{d-1} \le z < u_1^k$, then we consider $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{d-1}, z)$. Since it is always true that $z \le c^{u_1^k u_{k+1}\cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1^k\cdots u_{d-2}} \rceil - 1 = c^{u_1^k u_{k+1}\cdots u_{d-1}} - 2$, $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{d-1}, z)$ has a distinguishing c-labeling.

(iii) If $u_d < u_1^k$, then $u_d \leq 2^{u_1^k u_{k+1} \cdots u_{d-2}} - 2$. Thus, $(u_1, u_2, \ldots, u_{d-2}, u_d)$ has a distinguishing 2-labeling and so, by Lemma 4.1, $(u_1, u_2, \ldots, u_{d-2}, u_{d-1}, u_d)$ has one too.

Case 3: $(u_1 < u_2 = \cdots = u_k < u_{k+1} \leq \cdots \leq u_{d-1} \leq u_d)$, where $3 \leq k \leq d-1$

(i) If $u_2^{k-1} \leq u_{d-1}$, then $(u_1, u_{k+1}, \ldots, u_2^{k-1}, \ldots, u_{d-1}, u_d)$ has a distinguishing c-labeling if $u_d \leq c^{u_1 u_2^{k-1} \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2^{k-1} \cdots u_{d-2}} \rceil - 1$.

(ii) If $u_{d-1} < u_2^{k-1} \leq u_d$, then $(u_1, u_{k+1}, \ldots, u_{d-1}, u_2^{k-1}, u_d)$ has a distinguishing c-labeling if $u_d \leq c^{u_1 u_2^{k-1} \cdots u_{d-1}} - \lceil \frac{\log_c u_2^{k-1}}{u_{k+1} \cdots u_{d-1}} \rceil - 1$.

If $c^{u_1 u_2^{k-1} \cdots u_{d-1}} - \lceil \frac{\log_c u_2^{k-1}}{u_{k+1} \cdots u_{d-1}} \rceil - 1 < u_d \leq c^{u_1 u_2^{k-1} \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil - 1$, then switch and look at $2 = \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil + 1 \leq z < \lceil \frac{\log_c u_2^{k-1}}{u_{k+1} \cdots u_{d-1}} \rceil + 1 \leq k < u_2^{k-1}$.

(a) If $z < u_{d-1}$, then consider $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, z, \ldots, u_{d-1})$. Note that $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, z, \ldots, u_{d-1})$ has a distinguishing c-labeling if $u_{d-1} \leq c^{u_1 u_2^{k-1} \cdots u_{d-2} z} - \lceil \frac{\log_c \delta}{\varepsilon} \rceil - 1$, where $\varepsilon = u_1 u_2 \cdots u_{d-2}$ if $\delta = z$ and $\varepsilon = u_1 u_2 \cdots u_{d-3} z$ if $\delta = u_{d-2}$. Since, in either case, $\lceil \frac{\log_c \delta}{\varepsilon} \rceil \leq u_{d-1}$, it follows that $u_{d-1} \leq c^{u_1 u_2^{k-1} \cdots u_{d-2} z} - \lceil \frac{\log_c \delta}{\varepsilon} \rceil - 1$ always holds.

(b) If $u_{d-1} \leq z < u_2^{k-1}$, then look at $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{d-1}, z)$. Since $z \leq c^{u_1 u_2^{k-1} \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2^{k-1} \cdots u_{d-2}} \rceil - 1 = c^{u_1 u_2^{k-1} \cdots u_{d-1}} - 2$ always holds, $(u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{d-1}, z)$ has a distinguishing c-labeling by induction on u_d.

(iii) If $u_d < u_2^{k-1}$, then $u_d \leq 2^{u_2^{k-1} u_{k+1} \cdots u_{d-1}} - 2$. Thus $(u_2, u_3, \ldots, u_d)$ has a distinguishing 2-labeling and so, by Lemma 4.1, $(u_1, u_2, \ldots, u_d)$ has one too.

Case 4: $(u_1 \leq u_2 = \cdots = u_{d-1} = u_d)$ and $(u_1 = u_2 = \cdots = u_{d-1} < u_d)$

(i) For $(u_1 \leq u_2 = u_3 = \cdots = u_{d-1} = u_d)$, we note that $(u_2, u_3, \ldots, u_d)$ has a distinguishing 2-labeling and so, by Lemma 4.1, $(u_1, u_2, \ldots, u_d)$ has one too.

(ii) For $(u_1 = u_2 = \cdots = u_{d-1} < u_d)$, $u_1 < u_d < u_1^{d-2}$, we note that $u_d < u_1^{d-1}$. Therefore, since $(u_1, u_2, \ldots, u_{d-1})$ has a distinguishing 2-labeling, Lemma 4.1 implies that $(u_1, u_2, \ldots, u_{d-1}, u_d)$ also has a distinguishing 2-labeling.

(iii) If $u_1^{d-2} \leq u_d < u_1^{d-1}$, then (u_d, u_1^{d-1}) has a distinguishing 2-labeling since $u_1^{d-1} \leq 2^{u_1^{d-2}} - u_1^{d-1} - 1 \leq 2^{u_d} - u_d - 1$.

(iv) If $u_1^{d-1} \leq u_d \leq c^{u_1^{d-1}} - \lceil \log_c u_1^{d-1} \rceil - 1$, then (u_1^{d-1}, u_d) has a distinguishing c-labeling (since $u_d \leq c^{u_1^{d-1}} - \lceil \log_c u_1^{d-1} \rceil - 1$).

(v) Lastly, if $c^{u_1^{d-1}} - \lceil \log_c u_1^{d-1} \rceil \leq u_d \leq c^{u_1^{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1^{d-2}} \rceil - 1 = c^{u_1^{d-1}} - 2$, then we switch and look at $(z, u_1, u_2, \ldots, u_{d-1})$ or $(u_1, u_2, \ldots, u_{d-1}, z)$, where $2 \leq z \leq \lceil \log_c u_1^{d-1} \rceil < u_1^{d-1}$. In either case, it follows that $(u_1, u_2, \ldots, u_{d-1}, u_d)$ has a distinguishing c-labeling. $\square$

Lemma 4.3. For integers $c \geq 2$, $d \geq 3$ and $1 \leq u_1 \leq u_2 \leq \cdots \leq u_d$ with $u_d \geq c^{u_1 u_2 \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil + 1$, there is no distinguishing c-labeling of the $(u_1, u_2, \ldots, u_d)$ grid.

Proof. If $u_d > c^{u_1 u_2 \cdots u_{d-1}}$, then since the dimension d subgrids have size $(u_1, u_2, \ldots, u_{d-1})$ and there are greater than $c^{u_1 u_2 \cdots u_{d-1}}$ such subgrids, at least two of them must be identical. An automorphism switching two such positions in dimension d preserves labels. So, there is no distinguishing c-labeling in this case.

If $u_d = c^{u_1 u_2 \cdots u_{d-1}}$ and two dimension d subgrids are identical, apply the same automorphism as the previous paragraph. Otherwise, the set of dimension d subgrids is exactly the set of possible subgrids. Then, take any nontrivial automorphisms $\pi_1, \ldots, \pi_{d-1}$ in the other dimensions. This induces a permutation π_d of the dimension d subgrids. Then, $(\pi_1, \ldots, \pi_d)$ preserves labels. So, there is no distinguishing c-labeling in this case.

If $c^{u_1 u_2 \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil + 1 \leq u_d < c^{u_1 u_2 \cdots u_{d-1}}$, by the grid switching lemma there is a distinguishing c-labeling only if there is a distinguishing c-labeling of the grid of size $(u_1, u_2, \ldots, u_{d-1}, c^{u_d} - z)$, where $1 \leq z \leq \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil - 1$. Note that $u_{d-1} \geq z$ and that $z \cdot (u_1 \cdots u_{d-2}) < (\frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}}) \cdot (u_1 \cdots u_{d-2}) = \log_c u_{d-1}$. So, $c^{z \cdot (u_1 \cdots u_{d-2})} < c^{\log_c u_{d-1}} = u_{d-1}$. Hence, by the first paragraph, there is no distinguishing c-labeling. □

Theorem 4.4. For integers $d \geq 2$ and $1 \leq u_1 \leq \cdots \leq u_d$ let $s = u_1 u_2 \cdots u_{d-1}$ and $c = \lceil u_d^{1/s} \rceil$. Except for $d = 2$ with $u_1 = u_2 \in \{2, 3\}$ and for $d = 3$ with $u_1 = u_2 = u_3 = 2$ we have that the grid and array of size $(u_1, u_2, \ldots, u_d)$ have a distinguishing c-labeling if $u_d \leq c^{u_1 u_2 \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil - 1$. They have no distinguishing c-labeling if $u_d \geq c^{u_1 u_2 \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil + 1$.

Proof. If the array has a distinguishing c-labeling, so does the grid. If the grid has no distinguishing c-labeling, neither does the array. The result then follows immediately from Lemmas 4.2 and 4.3. □

In the remaining case, $u_d \geq c^{u_1 u_2 \cdots u_{d-1}} - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil$, the theorem gives no information. However, in these cases, we can apply the switching lemma to check. We again may need to recursively check the new sizes. However, the number of such steps will be at most the iterated logarithm (base c) of u_d.

5 Distinguishing numbers

In this section, we take the results of the previous section in order to determine (up to two possible values) the distinguishing numbers for arrays and grids.

Theorem 4.4 was stated from the perspective of whether or not a distinguishing c-labeling exists for given c and $(u_1, u_2, \ldots, u_d)$. For given $(u_1, u_2, \ldots, u_d)$, the distinguishing number (for a grid or array) is the smallest c such that the condition for a distinguishing labeling in Theorem 4.4 is satisfied. Determining this c we get the following result for distinguishing numbers.

Corollary 5.1. For integers $d \geq 2$ and $1 \leq u_1 \leq \cdots \leq u_d$, let $s = u_1 u_2 \cdots u_{d-1}$ and $c = \lceil u_d^{1/s} \rceil$. Except for $d = 2$ with $u_1 = u_2 \in \{2, 3\}$ and for $d = 3$ with $u_1 = u_2 = u_3 = 2$, we have $D(Grid(u_1, u_2, \ldots, u_d)) = D(Array(u_1, u_2, \ldots, u_d)) \in \{c, c+1\}$.

It is c if $u_d \leq c^s - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil - 1$ and $c+1$ if $c^s - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil + 1 \leq u_d$. When $u_d = c^s - \lceil \frac{\log_c u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil$, it is c if the grid of size $(u_1, u_2, \ldots, c^s - u_d)$ has a distinguishing c-labeling and $c+1$ otherwise. Hence, we can recursively determine the value in this case.

Proof. It is easy to check that for $k < c$, $u_d \geq k^s - \lceil \frac{\log_k u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil + 1$ and that $u_d \leq (c+1)^s - \lceil \frac{\log_{c+1} u_{d-1}}{u_1 u_2 \cdots u_{d-2}} \rceil - 1$. Hence, the smallest c such that there is a distinguishing c-labeling is c or $c+1$. The exact values follow immediately from Theorem 4.4. The comment about the recursion follows from the switching lemma. □

We finish with a few comments and examples. The distinguishing numbers for the exceptional cases in the corollary are 2 for grids and 3 for arrays as noted earlier. For the special cases when we use recursion, it is possible that the smaller grid/array has a distinguishing number smaller than c. However, what matters is whether or not the reduced case has a distinguishing c-labeling. Consider the following sizes for examples with $d = 3$: $(3, 5^6 - 2, 5^{3\cdot(5^6-2)} - 2)$, $(3, 5^6 - 1, 5^{3\cdot(5^6-1)} - 2)$, $(3, 5^6, 5^{3\cdot 5^6} - 2)$. In each case, $c = 5$ and we are on the boundary where we need to apply recursion. By the switching lemma, we look at (respectively) 5-labelings for sizes $(2, 3, 5^6 - 2)$, $(2, 3, 5^6 - 1)$, $(2, 3, 5^6)$. In the first case, the conditions give a 5-labeling and in the last case they do not. So $D(Grid(3, 5^6 - 2, 5^{3\cdot(5^6-2)} - 2) = 5$ and $D(Grid(3, 5^6, 5^{3\cdot 5^6} - 2) = 6$. In the middle case we again are on a boundary and apply the switching lemma one more time, looking at size $(1, 2, 3)$. Here, there is a distinguishing 5-labeling so we get $D(Grid(3, 5^6 - 1, 5^{3\cdot(5^6-1)} - 2) = 5$. It happens also that $D(Grid(2, 3, 5^6 - 1) = 5$, but it could have been lower. Note that $D(Grid(1, 2, 3) = 2$, but the fact that there is a distinguishing 2-labeling does not help for size $(2, 3, 5^6 - 1)$ as we need at least 5 labels for this size.

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