Math 163 Introductory Seminar - Lehigh University - Spring 2008 - Assignment 1 Solutions
Due Wednesday January 23

1. Let W be a set of men and M a set of women (with the same number of men and women, $|W| = |M| = n$) and E a set of pairs (w, m) with $w \in W$ and $m \in M$.

If there are subsets $R \subseteq W$ and $S \subseteq M$ such that $|R| + |S| < n$ and every pair in E contains at least one member of $R \cup S$ (that is, for each $(w, m) \in E$ either $w \in R$ or $m \in S$ or both), then there is no matching of the men and women with each pair from E. The marriage theorem shows that the converse also holds: if there is no matching of the men and women then there are R and S as described in the previous sentence.

Another condition is as follows: If there is a set T of women who 'like' strictly less than $|T|$ men then there is no matching of the men and women. More formally, if there is $T \subseteq W$ such that $|\{m|(w, m) \in E$ for some $w \in T\}| < |T|$ then there is no matching of the men and women. Use the marriage theorem to prove that the converse also holds: if there is no matching of men and women then there is a set T as described in the previous sentence.

If there is no matching then by the marriage theorem there are subsets $R \subseteq W$ and $S \subseteq M$ such that $|R| + |S| < n$ and for each $(w, m) \in E$ either $w \in R$ or $m \in S$ or both. So, if $(w, m) \in E$ and $w \in W - R$ then $m \in S$. Thus $\{m|(w, m) \in E \text{ for some } w \in W - R\} \subseteq S$. Then, using $|S| < n - |R|$ and $|W - R| = n - |R|$ we have $|\{m|(w, m) \in E \text{ for some } w \in W - R\}| \leq |S| < n - |R| = |W - R|$ and so $T = W - R$ give the desired set.

2. Prove by induction that the Fibonacci numbers satisfy the following formula:
$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$.

We prove that the formula is correct using mathematical induction. Since $F_0 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^0 + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^0 = \frac{1}{\sqrt{5}} + \frac{-1}{\sqrt{5}} = 0$ and $F_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^1 + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\sqrt{5} = 1$ the formula holds for $n = 0$ and $n = 1$. For $n \geq 2$, by induction

$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&= \left[\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] + \left[\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right] \\
&= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} + 1\right) \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} + 1\right) \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \\
&= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^2 \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^2 \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \\
&= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n
\end{aligned}$$

here we have also used $\frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} = \frac{1+2\sqrt{5}+5}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^2$ and similarly $\frac{1-\sqrt{5}}{2} + 1 = \left(\frac{1-\sqrt{5}}{2}\right)^2$. Hence by induction the formula holds for all $n = 0, 1, \ldots$.

3. Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

When $n = 1$ we have $\sum_{i=1}^{1} i^2 = 1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$ so the formula holds for $n = 1$. By induction we may assume $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} = \frac{(n-1)(n)(2n-1)}{6}$. Then $\sum_{i=1}^{n} i^2 = (\sum_{i=1}^{n-1} i^2) + n^2 = \frac{(n-1)(n)(2n-1)}{6} + \frac{6n^2}{6} = \frac{n[(n-1)(2n-1)+6n]}{6} = \frac{n[2n^2-3n+1+6n]}{6} = \frac{n(2n^2+3n+1)}{6} = \frac{n(n+1)(2n+1)}{6}$. Hence, by induction the formula holds for all $n = 1, 2, \ldots$.