Notes for Math 446, Fall 2004 Lehigh University: An alternate proof of Tutte's Matching Theorem. This follows the proof in exercises 6.2.41 - 6.2.43.

Recall Tutte's Theorem: A graph G has a perfect matching if and only if $odd(G-S) \leq |S|$ for all $S \subseteq V(G)$.

A graph is factor critical if every subgraph obtained by deleting a single vertex has a perfect matching.

Proof: Let G be a graph such that $odd(G-S) \leq |S|$ for all $S \subseteq V$. Let T be a maximal subset such that $odd(G-T) = |T|$. Such a subset exists since equality holds for the emptyset: with $S = \emptyset$ we have $|S| = 0$ and since $0 \leq odd(G-S) \leq |S|$ we must have equality. Note that every component of $G-T$ is odd: If $G-T$ has an even component C then let v be a leaf of a spanning tree of C. Then the odd components of $G-(T \cup \{v\})$ are the odd components of $G-T$ along with $C-\{v\}$. So $|T \cup \{v\}| = |T| + 1 = o(G-T) + 1 = o(G-T \cup \{v\})$ contradicting the maximality of T. Thus $G-T$ has no even components. If $T = \emptyset$, Tutte's condition gives no odd components and the previous observation gives no even components so G must be empty. So we may assume $T \neq \emptyset$.

Next we show that every component of $G-T$ is factor critical. From above all components are odd. Since $|C-x|$ is even, in $C-x$, $o((C-x)-S)$ must have the same parity as $|S|$. Hence if $o((C-x)-S) > |S|$ we must have $o((C-x)-S) \geq |S|+2$. Then consider $T \cup S \cup \{x\}$ in G. Deleting this gives the odd components of G other than C along with the odd components of $(C-x)-S$. So $|T \cup S \cup \{x\}| = |T| + 1 + |S| \leq o(G-T) + 1 + (o((C-x)-S) - 2) = o(G-T) + o((C-x)-S) - 1 = o(G-(T \cup S \cup \{x\}))$, contradicting the maximality of T. Thus, by induction there is a perfect matching in $C-x$.

Let H be a bipartite graph with partite sets T and $\mathbf{C}$ where $\mathbf{C}$ is the set of components of $G-T$. For $t \in T$ and $C \in \mathbf{C}$ put an edge between t and C in H if and only if $N_G(t)$ contains a vertex of C. We will show that H has a matching saturating $\mathbf{C}$ using Hall's Theorem. For $A \subseteq \mathbf{C}$ let $B = N_H(A)$. The elements of A are odd components of $G-B$ hence $|A| \leq o(G-B)$. Since Tutte's condition yields $o(G-B) \leq |B|$ we have $|N_H(A)| \geq |A|$.

Now, by Hall's Theorem and the previous paragraph, H has a matching that saturates $\mathbf{C}$. This matching yields $o(G-T) = |T|$ pairwise disjoint edges from odd components of $G-T$ to T. These are all of the components of $G-T$. These edges saturate one vertex from each component of $G-T$. Since the components are factor critical the vertices remaining in each component of $G-T$ are saturated by a perfect matching. The union of these matchings along with the pairwise disjoint edges above is a perfect matching in G. $\square$

The above proof actually nearly shows (with a little extra work for graphs with no perfect matching) the Gallai-Edmonds decomposition: There is a partition of the vertex set of G into parts $R_1, R_2, \ldots, R_l, T, C_1, C_2, \ldots, C_k$ such that $k = |T| + def(G)$. The even components of $G-T$ are the R_i and the odd components the C_j. The C_j are factor critical. Every maximum matching in G consists of perfect matchings in the R_i, $|T|$ edges between T and $|T|$ of the C_i and perfect matchings in $C_j - x_j$ for some x_j in C_j. Each vertex of each C_j is unsaturated in some maximum matching and all vertices in T and the R_i are saturated in all maximum matchings.