Linear Programming Duality Theorem from the Theorem of the Alternative for Inequalities:

(Notes for Math 446, Fall 2004 Lehigh University.)

We will assume the Theorem of the Alternative for Inequalities in the following form:

Exactly one of the following holds:

(I) $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$
(II) $\boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{yb} < 0$ has a solution $\boldsymbol{y}$

and use this to prove the following duality theorem for linear programming.

In what follows we will assume that A is an $m \times n$ matrix, $\boldsymbol{c}$ is a length n row vector, $\boldsymbol{x}$ is a length n column vector of variables, $\boldsymbol{b}$ is a length m column vector and $\boldsymbol{y}$ is a length n row vector of variables. We will use $\boldsymbol{0}$ for zero vectors, 0 for zero matrices, and I for identity matrices where appropriate sizes will be assumed and clear from context.

We will consider the following *primal* linear programming problem $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ and its *dual* $\min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$. (It can be shown that we can use maximum instead of supremum and minimum instead of infimum as these values are attained if they are finite.)

The primal is *feasible* if the polyhedron $\{x | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, called the *feasible region*, is non-empty and *infeasible* otherwise. Similarly, the dual is feasible if $\{y | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$ is non-empty. The primal is *unbounded* if the problem is feasible and the maximum does not exist and the dual is unbounded if it is feasible and the minimum does not exist.

Weak Duality Theorem of Linear Programming: If both the primal and dual are feasible then $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} \leq \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$.

Proof: For any feasible $\boldsymbol{x}^*$ and $\boldsymbol{y}*$ we have

$$\boldsymbol{cx}^* \leq (\boldsymbol{y}^*A)\boldsymbol{x}^* = \boldsymbol{y}^*(A\boldsymbol{x}^*) \leq \boldsymbol{y}^*\boldsymbol{b}$$

where the first inequality follows since $\boldsymbol{x}^* \geq \boldsymbol{0}$ and $\boldsymbol{y}^*A \geq \boldsymbol{c}$ and the second inequality follows since $\boldsymbol{y}^* \geq \boldsymbol{0}$ and $A\boldsymbol{x}^* \leq \boldsymbol{b}$. □

Strong Duality Theorem of Linear Programming: If both the primal and dual are feasible then $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$.

Proof: By weak duality we have max $\leq$ min. Thus it is enough to show that there are primal feasible $\boldsymbol{x}^*$ and dual feasible $\boldsymbol{y}^*$ with $\boldsymbol{cx}^* \geq \boldsymbol{y}^*\boldsymbol{b}$. We get this if $\boldsymbol{x}^*, \boldsymbol{y}^*$ is a feasible solution to

$$A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{cx} \geq \boldsymbol{yb}. \tag{1}$$

We can write (1) as $A'\boldsymbol{x}' \leq \boldsymbol{b}', \boldsymbol{x}' \geq \mathbf{0}$ where

$$A' = \left[\begin{array}{c|c} A & 0 \\ \hline -\boldsymbol{c} & \boldsymbol{b}^T \\ \hline 0 & -A^T \end{array}\right] \text{ and } \boldsymbol{x}' = \left[\begin{array}{c} \boldsymbol{x} \\ \hline \boldsymbol{y}^T \end{array}\right] \text{ and } \boldsymbol{b}' = \left[\begin{array}{c} \boldsymbol{b} \\ \hline 0 \\ \hline -\boldsymbol{c}^T \end{array}\right] \tag{2}$$

By the Theorem of the Alternative for Inequalities if (2) has no solution then

$$\boldsymbol{y}'A' \geq \mathbf{0}, \boldsymbol{y}' \geq \mathbf{0}, \boldsymbol{y}'\boldsymbol{b}' < 0 \tag{3}$$

has a solution. Writing

$$\boldsymbol{y}' = \left[\begin{array}{c|c|c} \boldsymbol{r} & s & \boldsymbol{t} \end{array}\right]$$

(3) becomes

$$\boldsymbol{r}A - s\boldsymbol{c} \geq \mathbf{0}, s\boldsymbol{b}^T - \boldsymbol{t}A^T \geq 0, \boldsymbol{r} \geq \mathbf{0}, s \geq 0, \boldsymbol{t} \geq \mathbf{0}, \boldsymbol{r}\boldsymbol{b} - \boldsymbol{t}\boldsymbol{c}^T < 0. \tag{4}$$

If we show that (4) has no solution then (1) must have a solution and we will be done.

Observe that s is a scalar (a number) as are $\boldsymbol{r}\boldsymbol{b}$ and $\boldsymbol{t}\boldsymbol{c}^T$. Assume that (4) has a solution $\boldsymbol{r}^*, s^*, \boldsymbol{t}^*$ and reach a contradiction.

Case 1: $s^* = 0$. Then $\boldsymbol{r}^*A \geq \mathbf{0}, \boldsymbol{r}^* \geq \mathbf{0}$. By the Theorem of the Alternative and since $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}$ has a solution we must have $\boldsymbol{r}^*\boldsymbol{b} \geq 0$. Then, with $\boldsymbol{r}^*\boldsymbol{b} - \boldsymbol{t}^*\boldsymbol{c}^T < 0$ we must have $\boldsymbol{t}^*(-\boldsymbol{c}^T) < 0$. Now, $\boldsymbol{t}^*(-A^T) \geq \mathbf{0}, \boldsymbol{t}^* \geq \mathbf{0}, \boldsymbol{t}^*(-\boldsymbol{c}^T) < 0$. By the Theorem of the Alternative there is no solution to $(-A^T)\boldsymbol{z} \leq (-\boldsymbol{c}^T), \boldsymbol{z} \geq \mathbf{0}$. This contradicts dual feasibility (taking as $\boldsymbol{z}$ the transpose of any dual feasible solution).

Case 2: $s^* \neq 0$. Let $\boldsymbol{r}' = \boldsymbol{r}^*/s^*$ and $\boldsymbol{t}' = (\boldsymbol{t}^*/s^*)^T$. Then, from (4) we have

$$\boldsymbol{r}'A \geq \boldsymbol{c}, A\boldsymbol{t}' \leq \boldsymbol{b}, \boldsymbol{r}' \geq \mathbf{0}, \boldsymbol{t}' \geq \mathbf{0}, \boldsymbol{r}'\boldsymbol{b} - \boldsymbol{c}\boldsymbol{t}' < 0.$$

But $\boldsymbol{r}'\boldsymbol{b} - \boldsymbol{c}\boldsymbol{t}' < 0$ implies $\boldsymbol{c}\boldsymbol{t}' > \boldsymbol{r}'\boldsymbol{b}$ contradicting weak duality. Thus, (4) has no solution and hence (1) has solution. $\Box$

We can in fact easily show that if either the primal or the dual has a finite optimum then so does the other. (This will be a homework problem.) Weak duality shows that if the primal or dual is unbounded then the other must be infeasible. Thus there are four possibilities for a primal-dual pair: both infeasible; primal unbounded and dual infeasible; dual unbounded and primal infeasible; both primal and dual with equal finite optima.

Complementary Slackness Theorem: If $\boldsymbol{x}^*$ is optimal for the primal and $\boldsymbol{y}^*$ is optimal for the dual then for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$:

(i) Either $(Row_i A)\boldsymbol{x^*} = b_i$ or $y_i^* = 0$
(ii) Either $\boldsymbol{y}^*(Column_j A) = c_j$ or $x_j^* = 0$.

Proof: From strong duality $\boldsymbol{cx}^* = \boldsymbol{y}^*\boldsymbol{b}$ and thus the inequalities in the proof of weak duality must hold with equality. Writing out $\boldsymbol{y}^*(A\boldsymbol{x}^*) - \boldsymbol{y}^*\boldsymbol{b} = 0$ as a sum we get $\displaystyle\sum_{i=1}^{m} y_i^* \left((Row_i A)\boldsymbol{x^*} - b_i\right) = 0$. Since $\boldsymbol{y}^* \geq \boldsymbol{0}$ and $A\boldsymbol{x}^* - \boldsymbol{b} \geq \boldsymbol{0}$ each term is non-negative. Hence, for equality each term must be zero and (i) follows. We get (ii) in a similar manner. □

The terminology in the previous theorem comes from slack variables. Using slack variables $\boldsymbol{s}$, $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ is equivalent to $A\boldsymbol{x} + I\boldsymbol{s} = \boldsymbol{b}, \boldsymbol{s} \geq \boldsymbol{0}, \boldsymbol{x} \geq \boldsymbol{0}$. The i^{th} entry of $\boldsymbol{s}$ records the slack (the gap) between the i^{th} entry of $A\boldsymbol{x}$ and the u^{th} entry of $\boldsymbol{b}$. The complementary slackness theorem (part (i)) states that for each i, either the slack in the i^{th} inequality is zero (equality holds) or the corresponding dual variable is zero.