Solutions to Homework 9
Combinatorics (Math 446) Fall 2004 Lehigh University

1.1.8 (a) The number of even elements in $[n]$ is $\lfloor n/2 \rfloor$. So there are $2^{\lfloor n/2 \rfloor}$ subsets avoiding odd elements and $2^n - 2^{\lfloor n/2 \rfloor}$ contain at least on eodd number.

(b) If $\{a_1, a_2, \ldots, a_k\}$ is a subset of $[n]$ with no consecutive integer we may assume that the labels are such that $a_1 < a_2 < \cdots a_k$. Indeed $a_j > a_{j-1}+1$. Define $b_1, b_2, \ldots, b_k$ by $b_i = a_i - i + 1$ for $i = 2, 3, \ldots, k$. Observe that the b_i are distinct (since $a_j > a_{j-1} + 1$) and correspond to a subset of $n-k+1$ and that this process can be reversed so that we have a bijection between k subsets of $[n]$ with no consecutive integer and k subsets of $n-k+1$. So the answer is $\binom{n-k+1}{k}$.

(c) If and element is in A_i then it is in A_j for all $j \geq i$. Thus we specify the lists by specifying for each element the index of the first subset for which it appears. For $A_0 \subset A_1 \subset \cdots \subset A_n$ in order to maintain $\subset$ at least one element can be added for each of the n $\subset$'s. Since there are n elements we must have $A_0 = \emptyset$ and one element added for each other subscript yielding $n!$. For $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$ there are $n+1$ subscripts for when each element first appears or it may not appear at all. Thus the list corresponds to a list of length n from $n+2$ symbols. So the answer is $(n+2)^n$.

1.1.22 Assume poker hands here. So in two-pair the remaining card is a different rank from the ones in the pair, in three-of-a-kind the remaining two cards are different ranks (to avoid a full house) and different from the rank of the three (to avoid four-of-a-kind), a straight is not a straight flush and a flush is not a straight flush. There are 123,552 two-pair hands, 54,912 three-of-a-kind, 10,200 straights and 5,108 flushes. With $\binom{52}{5} = 2,598,960$ hands the probabilities are respectively .047539, .021129, .003925, .001965.

To get the counts:

two-pair: $\binom{13}{2}$ ways to pick the ranks for the pair and $\binom{4}{2}\binom{4}{2}$ to pick the suits for the two cards of each of the pairs, 11 choices (different from the pairs) for the rank of the remaining card and 4 choices for its suit - total is $\displaystyle\binom{13}{2}\binom{4}{2}\binom{4}{2}(11)(4) = 123,552$

three-of-a-kind: 13 ways to pick the rank for the three and $\binom{4}{3} = 4$ ways to pick the suits of the three cards, $\binom{12}{2}$ ways to pick two different ranks from the remaining 12 ranks for the other two cards and (4)(4) ways to pick their suits - total is $(13)(4)\displaystyle\binom{12}{2}(4)(4) = 54,912$

straights: 10 possible ranks to start with and $4^5 - 4$ ways to pick the suits of the five cards (the minus 4 omits the 4 cases of straight flushes where all suits are the same) - total is $10(4^5 - 4) = 10,200$

flushes: 4 choices for the flush suit and $\binom{13}{5}$ ways to pick five cards of this suit and then omit the (10)(4) straight flushes - total $(4)\displaystyle\binom{13}{5} - (10)(4) = 5,108$

1.1.33 There are several simple ways to count the number of placements of m distinct flags on r flagpoles. Here are two similar ones:

Version 1: If the labels on the flags are removed we have a placement of m indistinct flags on r flagpoles. The number of such placements is the number of non-negative integral solutions to $\sum_{i=1}^{r} x_i = m$ with the x_i indicating how many flags are on pole i. (This then equals the number of m element multisets of an r set.) There are $\binom{m+r-1}{m}$ such solutions. For each such unlabelled placement there are $m!$ ways to label the flags. Hence we get $\binom{m+r-1}{m}m! = (m+r-1)!/(r-1)! = r^{(m)}$

Version 2: Each placement of flags on the poles corresponds to an anagram of with m distinct symbols (the labels on the flags) along with one symbol $|$ repeated $r-1$ times. The lists between the $|$ give the placements of the flags on each of the poles. There are $(m+r-1)!/(r-1)! = r^{(m)}$ such anagrams.

To show $(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}$ consider placing n distinct flags onto a set of $x+y$ flagpoles. We count this in two ways. Directly we get $(x+y)^{(n)}$ from the first part of the problem. Alternatively, note that there are k flags on the first x poles where k can be $0, 1, 2, \ldots, n$. For each such k we choose the k flags on the first x poles in $\binom{n}{k}$ ways and then place the flags on the first x poles in $x^{(k)}$ ways and place the remaining $n-k$ flags on the last y poles in $y^{(n-k)}$ ways. So for each k we have $\binom{n}{k} x^{(k)} y^{(n-k)}$ ways of placing the flags. Summing over k we get the identity.

1.1.36 The period of a sequence of length n is the smallest k such that the sequence can be written as n/k copies of an identical string of length k. Call two sequences equivalent if they are cyclic shifts of each other. It is straightforward to check that sequences in the same equivalence class have the same period and the number in a class is equal to that period. Since n/k is an integer the period divides the length. Thus, when the length p is a prime the only possible periods are 1 and p. The are a period 1 strings (these are strings with every term the same). The remaining $a^p - a$ sequences are divided into equivalence classes each of size p. So p must divide $a^p - a$.

1.2.2 Both sides count the number of subset pairs A, B of $[m+n]$ with $A \subset B$ and $|A| = k$ and $|B| = m+k$. For the left side there are $\binom{m+n}{m+k}$ ways to select B and then $\binom{m+k}{k}$ ways to select A as a subset of B. For the right side there are $\binom{m+n}{m}$ ways to select $B - A$ and $\binom{n}{k}$ ways to select A from the remaining n elements.

1.2.11 Observe that the $j = m$ term in the sum is 0. Then we obtain a recursion for $f(m)$

$$f(m) = \sum_{j=1}^{m-1} (m-j) 2^{j-1} = \sum_{j=1}^{m-1} (m-j-1) 2^{j-1} + \sum_{j=1}^{m-1} 2^{j-1} = f(m-1) + 2^{m-1} - 1$$

Here we have used $\sum_{j=1}^{m-1} 2^{j-1} = \sum_{i=0}^{m-2} 2^i = 2^{m-1} - 1$. Note that $f(1) = 0$. Then

$$\begin{aligned}
f(m) &= f(m-1) + (2^{m-1} - 1) \\
&= f(m-2) + (2^{m-2} - 1) + (2^{m-1} - 1) \\
&\vdots \quad \vdots \\
&= f(m-i) + (2^{m-i} - 1) + \cdots + (2^{m-j} - 1) + \cdots + (2^{m-1} - 1) \\
&\vdots \quad \vdots \\
&= f(1) + (2^{m-(m-1)} - 1) + \cdots + (2^{m-j} - 1) + \cdots + (2^{m-1} - 1) \\
&= 0 + \textstyle\sum_{j=1}^{m-1} 2^{m-j} - \sum_{j=1}^{m-1} 1 \\
&= (2^m - 2) - (m-1) \\
&= 2^m - m - 1
\end{aligned}$$

Once we know the formula (obtained for example as above or by some other means) we can easily prove its correctness by induction and the recursive formula we got first. (The steps above actually prove it also, so wouldn't be necessary. However it is useful to see how one might arrive at the formula.) The basis for the induction is easy to check $0 = f(1) = 2^1 - 1 - 1$. Then, by the recursion and by induction

$$f(m) = f(m-1) + (2^{m-1} - 1) = (2^{m-1} - (m-1) - 1) + (2^{m-1} - 1) = 2 \cdot 2^{m-1} m - 1 = 2^m - m - 1$$

and by induction the formula is correct.

For a combinatorial proof count the subsets of size at least 2 from $[m] = \{1, 2, \ldots, m\}$. This is $2^m - 1$ as there are m size 1 subsets and 1 size 0 subset omitted from all of the 2^m subsets. For the sum, partition the subsets (of size at least 2) based on the second largest element j and sum over possible values of j. If the second largest element is j then there are 2^{j-1} ways to pick the elements other than the largest two and $m - j$ choices for the largest element for a total of $(m-j)2^{j-1}$ subsets of size at least two with largest element j.

1.2.12 (a) $\binom{2n}{n}$ is the number of size n subsets of $[2n] = \{1, 2 \ldots, 2n\}$. Alternatively, the size n subsets of $[2n]$ consist of complementary pairs $S, \overline{S}$. Exactly one of each pair does not contain $2n$ and there are $\binom{2n-1}{n-1}$ such sets and hence $2\binom{2n-1}{n-1}$ pairs.

(b) Both sides count the number of pairs R, S where $R \subseteq S \subseteq [n]$ and $|R| = l$. For the right side we select R in $\binom{n}{l}$ ways and for each choice there are 2^{n-l} ways to select $S - R$ from $[n] - R$. For the left side sum over all possible sizes for S. Select S in $\binom{n}{k}$ ways and then select R from S in $\binom{k}{l}$ ways.

(c) Consider all length n strings with entries from $[q]$ except for the string of all q's. Call two strings equivalent if they are identical except for the last entry that is not q. There are $q - 1$ strings in each equivalence class and hence $(q^n - 1)/(q - 1)$ equivalence classes. The left side also counts this. The last non q entry can be in location k for $k = 1, 2, \ldots, n$, for each the first $k - 1$ terms can be selected in q^{k-1} ways and each such choice completed with a non q term followed by all q's determines an equivalence class. Summing over possible choices of k gives the left side.

(d) There is a typo; the upper limit in both sums should be n.

Both sides count the number of size 3 subsets of $[n+1]$. For the sum on the right partition the sets based on the largest element. It can be $i+1$ for $i = 2, 3, \ldots, n$. For each such choice there are $\binom{i}{2}$ ways to pick the smaller two elements. Summing gives the right side. (Note that $\binom{1}{2} = 0$ so we can add $i = 1$ to the sum.) For the sum on the left partition based on the middle element (in terms of value). It can be $i+1$ for $i = 1, 2, \ldots, n$. For each such choice there are i choices for the smaller element and $(n+1) - (i+1) = n - i$ choices for the larger element. Summing give the left side.

1.3.4 The arrangement of people corresponds to a $2 \times n$ array A with entries $1, 2, \ldots, 2n$ (corresponding to height order) increasing in rows and columns; $a_{1j} < a_{1j'}$ and $a_{2j} < a_{2j'}$ for $j < j'$ and $a_{1j} < a_{2j}$ for $j = 1, 2, \ldots, n$. Given such an array, construct a list $b_1, b_2, \ldots, b_{2n}$ by letting $b_i = 1$ if i appears in row 1 and $b_i = 0$ if i appears in row 2. So the b_i sequence has $2n$ terms of which half are one and half are zero. If $b_i = 1$ and i appears in $a_{1j'}$ then $a_{1j} < a_{1j'} < a_{1j''}$ and $a_{1j'} < a_{2j'} < a_{2j''}$ for for $j < j' < j''$. So $1, 2, \ldots, i-1$ appear in columns before column j' and all entries in row 1 are of this type. Thus we have $j' - 1$ ones appearing before b_i and at most $j' - 1$ zeros appearing before b_i. If $b_i = 0$ and i appears in $a_{2j'}$ then $a_{1j} < a_{1j'} < a_{2j'}$ and $a_{2j} < a_{2j'} < a_{2j''}$ for for $j < j' < j''$. So $1, 2, \ldots, i-1$ appear either in row 1 or before column j' in row 2 and all entries in the first j' columns of row one are of this type. Thus we have at least j' ones and at most $j' - 1$ zeros preceding b_i. In each case we set that the initial segment condition for ballot sequences is satisfied. This map can be reversed: given a ballot sequence, $b_1, b_2, \ldots, b_{2n}$ let $a_{1,j}$ be the subscript on the j^{th} one and $a_{2,j}$ be the subscript on the j^{th} zero. It is easy to see that this is the inverse map, so to establish that we have a bijection we need to check that the inverse images are arrays that increase in rows and columns. The definition makes them increase in rows and the initial segment property makes them increase on columns.

1.3.23 A sequence in which A is always ahead must begin AA. Deleting the initial A yields a sequence with $a - 1$ A's for which A is never behind. Adding an A at the start of a sequence for which A is never behind yields a sequence in which A is always ahead. This is a bijection between between 'always ahead' sequences with a A's and 'never behind' sequences with $a - 1$ A's. Using the ballot problem (with $a - 1$ replacing a) and dividing by the total number $\binom{a+b}{a}$ of sequences we get the probability

$$\frac{\binom{(a-1)+b}{a-1} - \binom{(a-1)+b}{(a-1)+1}}{\binom{a+b}{a}} = \frac{a-b}{a+b}$$