Solutions to Homework 7
Combinatorics (Math 446) Fall 2004 Lehigh University

34. (18.1.2) Take a star $K_{1,t}$ with all vertex weights 1 except the center with weight 2. A maximum weight independent set has size t. The greedy algorithm picks the center vertex first and then no others can be picked, yielding an independent set with weight 2. The gap $t-2$ can be arbitrarily large as t can be.

35. (18.1.8) If a set contains at least two elements from any part it is dependent. Thus the circuits are the two element sets $\{x, y\}$ where both x and y are in the same part. If two circuits intersect they must be of the form $C_1 = \{x, y\}$ and $C_2 = \{x, z\}$ with x, y in the same part and with x, z in the same part. Then $(C_1 \cup C_2) - x = \{y, z\}$ is a circuit (and thus contains a circuit). This shows that weak elimination holds.

36. (18.1.28) (a) $B + e$ contains a unique circuit C. If $f \in C$ then $B + e - f$ contains no circuits, So it is independent and has size the same as B. So it is a base. If $B + e - f$ is a base, it does not contain a circuit and since C is contained in $B + e$ it must be the case that $f \in C$.

(b) We are given that $B' = B - f + e$ and $B'' = B' - f' + e' = B - f, f' + e, e'$ are bases. We need to assume that $e \neq e'$ and $f \neq f'$ or else the size of B'' is not correct to be a base. Since $B' - f' + e'$ is a base, e', f' are on the unique circuit C' formed by adding e' to $B' = B - f + e$. If $e \notin C'$ then $f' \in C' \subseteq B + e'$ and $B - f' + e'$ is a base. So $e \in C'$. Also, since $B - f + e$ is a base we have e, f are on the unique circuit C formed by adding e to B. C' and C are distinct as $e' \in C'$ and $e' \notin C$. By weak elimination there exists $C''' \subseteq (C \cup C') - e \subseteq ((B + e) \cup (B - f + e, e')) - e = B + e'$. If $f' \in C''$ then $B - f' + e'$ is a base. If $f' \notin C''$ then $C'' \subseteq B''$ a contradiction.