Solutions to Homework 6
Combinatorics (Math 446) Fall 2004 Lehigh University

28. Assume that the equilateral triangles have side length s. The heuristic starts with a minimum spanning tree. Since the distance between any two points is at least s, a minimum spanning tree has weight at least $s(2m-2)$. Thus the spanning tree consisting of the path $1, 2m-1, 2, 2m-2, 3, 2m-3, \ldots, m+2, m-1, m+1, m$ is minimum. With this, the only two odd degree vertices are 1 and m and the heuristic adds the edge between them with weight $s(m-1)$. The result is a Hamiltonian path, so there is no shortcutting and the heuristic produces a Hamiltonian path of length $s(2m-2+m-1) = s(3m-3)$. A minimum weight Hamiltonian tour has weight at least $s(2m-1)$ as each edge has weight at least s and there are $2m-1$ edges. Given ϵ, take $m > \frac{3}{4\epsilon} + \frac{1}{2}$ then it can be checked (with some straightforward algebra) that $-3/2 \geq -\epsilon(2m-1)$. Thus, for such m we have $3m-3 = (2m-1) + .5(2m-1) - 3/2 \geq (2m-1) + (.5-\epsilon)(2m-1)$ as needed.

29. (6.2.4) We give three different proofs:

Proof 1: Let M^* be a maximum matching and M maximal. M^* has $2|M^*|$ vertices as ends and M has $2|M|$. So at least $2|M^*| - 2|M|$ vertices of M^* are not ends of edges of M. If $|M| < |M^*|/2$ then $2|M^*| - 2|M| > |M^*|$ and since more than $|M^*|$ vertices are not covered by M, both ends of some edge of M^* are not covered by M. This edge could then be added to M contradicting maximality.

Proof 2: By maximality, the vertices that are ends of edges in M cover all edges (an edge not covered could be added to M). Thus there is a vertex cover of size at most $2|M|$. So $2|M| \geq \beta(G)$. Since also we have $\beta(G) \geq \alpha'(G)$ (weak duality) we get $2|M| \geq \alpha'(G)$.

Proof 3: Let M^* be a maximum matching and M maximal. Consider the symmetric difference $M \triangle M^*$. There are at least $|M^*| - |M|$ augmenting paths in this symmetric difference. Since M is maximal none of these paths consists of a single edge from M^*. Thus each augmenting path contains at least one edge from M and we get $|M| \geq |M^*| - |M| \Rightarrow |M| \geq |M^*|/2$.

30. (6.2.24) Show that Tutte's condition holds. This implies the existence of a 1-factor. Note that $G - S$ is connected for $|S| < r$ and in particular that G is connected. If $1 \leq |S| < r$ then $odd(G-S) \leq 1$ and we have $odd(G-S) \leq |S|$. Since G is connected and has even order $odd(G_S) = 0$ when $S = \emptyset$ and we have $odd(G_S) \leq |S|$ for $S = \emptyset$. For $|S| \geq r$ construct a bipartite graph H with parts $S = \{v_1, v_2, \ldots, v_s\}$ and the components $C_1, C_2, \ldots, C_t$ with $t = odd(G-S)$. Put an edge between v_i and C_j if there is at least one edge between v_i and a vertex of C_j. Since there is no $K_{1,r+1}$ the degree of each v_i in H is at most r. Since deleting fewer

than r vertices does not disconnect the graph, the degree of each C_j in H is at least r. If H has e edges, counting the edges in two ways we get $sr \geq e \geq tr$. So $s \geq t$, which is $|S| \geq odd(G_S)$. So Tutte's condition holds in all cases.

31. (6.3.6) (a) Consider the matrix

$$\begin{array}{ccccc}
4 & 4 & 4 & 3 & 6 \\
1 & 1 & 4 & 3 & 4 \\
1 & 4 & 5 & 3 & 5 \\
5 & 6 & 4 & 7 & 9 \\
5 & 3 & 6 & 8 & 3
\end{array}$$

We can find an initial cover by taking $u_i = \max_j row_i$ for all i, yielding

$$\begin{array}{c|ccccc|}
 & 0 & 0 & 0 & 0 & 0 \\
6 & 2 & 2 & 2 & 3 & 0 \\
4 & 3 & 3 & 0 & 1 & \underline{0} \\
5 & 4 & 1 & \underline{0} & 2 & 0 \\
9 & 4 & 3 & 5 & 2 & 0 \\
8 & 3 & 5 & 2 & \underline{0} & 5
\end{array}$$

where the underlined zeros correspond to a matching. If we let $R = \{\emptyset\}$ and $T = \{3, 4, 5\}$ the minimum ϵ of the uncovered elements is equal to 1. Thus, we decrease u_i by 1, for all i and increase v_3, v_4, v_5 by 1, yielding

$$\begin{array}{c|ccccc|}
 & 0 & 0 & 1 & 1 & 1 \\
5 & 1 & 1 & 2 & 3 & 0 \\
3 & 2 & 2 & \underline{0} & 1 & 0 \\
4 & 3 & \underline{0} & 0 & 2 & 0 \\
8 & 3 & 2 & 5 & 2 & \underline{0} \\
7 & 2 & 4 & 2 & \underline{0} & 5
\end{array}$$

Here, we let $R = \{\emptyset\}$ and $T = \{2, 3, 4, 5\}$. This gives $\epsilon = 1$, so we decrease u_i by 1 for all i and increase v_2, v_3, v_4, v_5 by 1, yielding the final solution

$$\begin{array}{c|ccccc|}
 & 0 & 1 & 2 & 2 & 2 \\
4 & \underline{0} & 1 & 2 & 3 & 0 \\
2 & 1 & 2 & \underline{0} & 1 & 0 \\
3 & 2 & \underline{0} & 0 & 2 & 0 \\
7 & 2 & 2 & 5 & 2 & \underline{0} \\
6 & 1 & 4 & 2 & \underline{0} & 5
\end{array}$$

where $c(u, v) = 29 = w(M)$.

(b) Consider the matrix

$$\begin{array}{ccccc}
7 & 8 & 9 & 8 & 7 \\
8 & 7 & 6 & 7 & 6 \\
9 & 6 & 5 & 4 & 6 \\
8 & 5 & 7 & 6 & 4 \\
7 & 6 & 5 & 5 & 5
\end{array}$$

We can find an initial cover by taking $u_i = \max_j row_i$ for all i, yielding

$$\begin{array}{c|ccccc|}
 & 0 & 0 & 0 & 0 & 0 \\
9 & 2 & 1 & \underline{0} & 1 & 2 \\
8 & \underline{0} & 1 & 2 & 1 & 2 \\
9 & 0 & 3 & 4 & 5 & 3 \\
8 & 0 & 3 & 1 & 2 & 4 \\
7 & 0 & 1 & 2 & 2 & 2
\end{array}$$

If we let $R = \{\emptyset\}$ and $T = \{1, 3\}$, we have $\epsilon = 1$, so we decrease u_i by 1 for all i and increase v_1, v_3 by 1, yielding

$$\begin{array}{c|ccccc|}
 & 1 & 0 & 1 & 0 & 0 \\
8 & 2 & 0 & \underline{0} & 0 & 1 \\
7 & 0 & 0 & 2 & \underline{0} & 1 \\
8 & 0 & 2 & 4 & 4 & 2 \\
7 & \underline{0} & 2 & 1 & 1 & 3 \\
6 & 0 & \underline{0} & 2 & 1 & 1
\end{array}$$

Then, if we let $R = \{\emptyset\}$ and $T = \{1, 2, 3, 4\}$, we have $\epsilon = 1$, so we decrease u_i by 1 for all i and increase v_1, v_2, v_3, v_4 by 1, yielding

$$\begin{array}{c|ccccc|}
 & 2 & 1 & 2 & 1 & 0 \\
7 & 2 & 0 & \underline{0} & 0 & 0 \\
6 & 0 & 0 & 2 & \underline{0} & 0 \\
7 & \underline{0} & 2 & 4 & 4 & 1 \\
6 & 0 & 2 & 1 & 1 & 2 \\
5 & 0 & \underline{0} & 2 & 1 & 0
\end{array}$$

Finally, we let $R = \{1, 2, 5\}$ and $T = \{1\}$, giving $\epsilon = 1$, and yielding the final solution

$$\begin{array}{c|ccccc|}
 & 3 & 1 & 2 & 1 & 0 \\
7 & 2 & \underline{0} & 0 & 0 & 0 \\
6 & 0 & 0 & 2 & \underline{0} & 0 \\
6 & \underline{0} & 1 & 3 & 3 & 0 \\
5 & 0 & 1 & \underline{0} & 0 & 1 \\
5 & 0 & 0 & 2 & 1 & \underline{0}
\end{array}$$

where $c(u,v) = w(M) = 36$.

(b) Consider the matrix

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 7 & 2 \\ 1 & 3 & 4 & 4 & 5 \\ 3 & 6 & 2 & 8 & 7 \\ 4 & 1 & 3 & 5 & 4 \end{array}$$

We can find an initial cover by taking $u_i = \max_j row_i$ for all i, yielding

$$\begin{array}{c|ccccc|} & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 8 & 2 & 1 & \underline{0} & 1 & 6 \\ 5 & 4 & 2 & 1 & 1 & \underline{0} \\ 8 & 5 & 2 & 6 & \underline{0} & 1 \\ 5 & 1 & 4 & 2 & 0 & 1 \end{array}$$

If we let $R = \{\emptyset\}$ and $T = \{3,4,5\}$, we have $\epsilon = 1$, so we adjust the cover and the excess graph, accordingly, yielding

$$\begin{array}{c|ccccc|} & 0 & 0 & 1 & 1 & 1 \\ 4 & 3 & 2 & 2 & 1 & 0 \\ 7 & 1 & 0 & \underline{0} & 1 & 6 \\ 4 & 3 & 1 & 1 & 1 & \underline{0} \\ 7 & 4 & 1 & 6 & \underline{0} & 1 \\ 4 & \underline{0} & 3 & 2 & 0 & 1 \end{array}$$

Then, if we let $R = \{2,4\}$ and $T = \{4,5\}$, we have $\epsilon = 1$, leading to the final solution

$$\begin{array}{c|ccccc|} & 0 & 0 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 & 1 & 0 \\ 7 & 1 & 0 & \underline{0} & 1 & 6 \\ 3 & 2 & 0 & 0 & 1 & \underline{0} \\ 6 & 3 & \underline{0} & 5 & 0 & 1 \\ 4 & \underline{0} & 3 & 2 & 0 & 1 \end{array}$$

where $c(u,v) = w(M) = 28$

32. (6.3.8) Create a bipartite graph with parts $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ and the weight on edge $u_i v_j$ equal to $\max\{0, x_i + y_j - t\}$. The weights are the overtime, if any, for the corresponding pairing of routes. Thus we solve the problem by finding a minimum weight perfect matching.

Label so that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. To show that a best solution is to pair the i^{th} shortest with the i^{th} longest we need to show that matching using edges u_iv_i for $i = 1, 2, \ldots, n$ is minimum. We will show this using induction. If $n = 1$ the result is trivial. For $n > 1$ if u_1v_1 is in the matching delete this edge and use induction on the remaining edges. We will show that there exists a minimum matching pairing u_1v_1 and the apply induction as in the previous sentence to establish the result. Assume that u_1v_j and v_1u_i are in the minimum matching with weight c^*. Switching these edges to match u_1v_1 and u_iv_j yields a matching with weight c such that $c^* - c = \max\{0, x_1+y_j-t\} + \max\{0, x_i+y_1-t\} - \max\{0, x_1+y_1-t\} - \max\{0, x_i+y_j-t\}$ (as all other weights remain unchanged). With $x_1 \leq x_i$ and $y_1 \geq y_j$ we get

Case 1: If $x_i + y_1 \leq t$ then each of $x_1 + y_1, x_i + y_j, x_1 + y_j$ is at most t. In this case the weights on all four edges are 0 and $c^* - c = 0$.

Case 2: $x_i + y_j \leq t$. Then $x_1 + y_j \leq t$ and the weights on edges u_iy_j and u_1y_j are 0. As also $x_1 + y_1 \leq x_i + y_1$ the weight on edge u_1v_1 is at most that on edge u_iy_1 and $c^* - c \geq 0$.

Case 3: $x_1 + y_1 \leq t$. Then $x_1 + y_j \leq t$ and the weights on edges u_1v_1 and u_1v_j are 0. As $x_i + y_j \leq x_i + y_1$ the weight on edge u_iv_j is at most that on edge u_iy_1 and $c^* - c \geq 0$.

Case 4: None of the above. Then The weights on edges u_1v_1, u_iv_j, u_iv_1 are respectively $x_1 + y_1 - t, x_i + y_j - t, x_i + y_1 - t$ and $c^* - c = (\max\{0, x_1 + y_j - t\}) + (x_i + y_1 - t) - (x_1 + y_1 - t) - (x_i + y_j - t) = (\max\{0, x_1 + y_j - t\}) - (x_1 + y_j - t) \geq 0$.

Thus in each case switching does not increase the weight and we get a minimum matching using the edge u_1v_1 and as noted above by induction the result follows.

33. (6.3.11) Form a bipartite graph with bipartition $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_{rs} | r = 1, 2, \ldots, k \text{ and } s = 1, 2 \ldots, k_r\}$. Put the weight on edge u_iv_{rs} to be t if seminar r is the t^{th} highest seminar on the list of student i. A minimum weight perfect matching is stable under this definition of stable. (Here we put student i in seminar r if u_i is matched to v_{rs} for some s.) If student i in seminar r and student i' in seminar r' want to switch then i prefers r' and i' prefers r so for any s, s', s'', s''' we have $weight(u_iv_{rs}) > weight(u_iv_{r's'})$ and also $weight(u_{i'}v_{r's''}) > weight(u_{i'}v_{rs'''})$. Thus the matching after the switch has lower weight, a contradiction.

Note - one can also check that a 'greedy' approach, going through the list of students and assigning the highest ranked seminar with an available slot will produce a stable matching.