Solutions to Homework 910
Combinatorics (Math 446) Fall 2004 Lehigh University

2.1.9 (a) For induction the basis is $\hat{F}_0 = 1 = \binom{0}{0}$ and $\hat{F}_1 = 1 = 1 + 0 = \binom{1}{0} + \binom{0}{1}$. Then using induction, Pascal's identity and the recurrence for $\hat{F}$ we get for $n \geq 2$

$$\begin{aligned}
\hat{F}_n &= \hat{F}_{n-1} + \hat{F}_{n-2} \\
&= \sum_{j=0}^{n-1} \binom{n-1-j}{j} + \sum_{k=0}^{n-2} \binom{n-2-k}{k} \\
&= \binom{n-1-0}{0} + \sum_{j=1}^{n-1} \binom{n-1-j}{j} + \sum_{k+1=1}^{n-1} \binom{n-1-(k+1)}{(k+1)-1} \\
&= 1 + 0 + \sum_{i=1}^{n-1} \left( \binom{n-1-i}{i} + \binom{n-1-i}{i-1} \right) \\
&= \binom{n-0}{0} + \binom{n-n}{n} + \sum_{i=1}^{n-1} \binom{n-i}{i} \\
&= \sum_{i=0}^{n} \binom{n-i}{i}
\end{aligned}$$

Combinatorially, $\hat{F}_n$ counts the number of $1,2$ sequences summing to $n$. Partition the sequences by the number of 2's in the sequence. If there are $i$ 2's then there are $n-i$ terms in the sequence, of which $i$ are 2's. So we pick the locations for the 2's in $\binom{n-i}{i}$ ways. Summing over $i$ gives the identity.

(b) For induction the basis is $\hat{F}_{0+2} = 2 = 1 + 1 = 1 + \hat{F}_0$. Then, using induction and the recurrence for $\hat{F}_n$ we get for $n \geq 1$

$$\hat{F}_{n+2} = \hat{F}_{n+1} + \hat{F}_n = (1 + \sum_{i=0}^{n-1} \hat{F}_i) + \hat{F}_n = 1 + \sum_{i=0}^{n} \hat{F}_i.$$

Combinatorially, $\hat{F}_{n+2}$ counts the number of $1,2$ sequences summing to $n+2$. There is one sequence of all 1's. Partition the remaining sequences by the first occurrence of a 2. If the first 2 appears after $i$ 1's then the sequence begins with $i$ 1's and a 2 with the rest of the sequence an arbitrary $1,2$ sequence summing to $(n+2)-i-2 = n-i$; there are $\hat{F}_{n-i}$ of these. Summing over $i$ we get $\hat{F}_{n+2} = 1 + \sum_{i=0}^{n} \hat{F}_{n-i} = 1 + \sum_{i=0}^{n} \hat{F}_i$.

2.2.4 The homogeneous part $a_n = 3a_{n-1} - 2a_{n-2}$ has characteristic polynomial $x^2 - 3x + 2 = (x-1)(x-2)$ with roots $1, 2$. From proposition 2.2.10 we see that there is a particular solution of the form $P(n)n^r 2^n$ where $r = 1$ and $P(n)$ is a polynomial with degree 1, i.e., a constant, call it $P$. Thus we get that our solution is of the form $a_n = c_1(1^n) + c_2(2^n) + Pn2^n = c_1 + c_2 2^n + Pn2^n$. From $a_0 = a_1 = 1$ we get $a_2 = 3a_1 - 2a_0 + 2^2 = 3(1) - 2(1) + 4 = 5$. Then using the general form we get $1 = a_0 = c_1 + c_2$; $1 = a_1 = c_1 + 2c_2 + 2P$ and $5 = a_2 = c_1 + 4c_2 + 8P$. This system of 3 equations in 3 unknowns has solution $c_1 = 5, c_2 = -4, P = 2$ and we get $a_n = 5 - 4 \cdot 2^n + 2n2^n = 5 - 4 \cdot 2^n + n2^{n+1}$.

Using generating functions we get

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} \left(3a_{n-1} - 2a_{n-2} + 2^n\right) x^n \\
&= 1 + x + 3x(\sum_{n-1=1}^{\infty} a_{n-1} x^{n-1}) - 2x^2(\sum_{n-2=0}^{\infty} a_{n-2} x^{n-2}) + \sum_{n=2}^{\infty} 2^n x^n \\
&= 1 + x + 3x(A(x) - 1) - 2x^2 A(x) + (2x)^2 \sum_{m=0}^{\infty} (2x)^m \\
&= 1 - 2x + (3x - 2x^2)A(x) + \frac{4x^2}{1-2x}
\end{aligned}$$

Then, solving for $A(x)$ we get

$$A(x) = \frac{1}{1 - 3x + 2x^2}\left(1 - 2x + \frac{4x^2}{1-2x}\right) = \frac{1}{(1-2x)(1-x)}\left(1 - 2x + \frac{4x^2}{1-2x}\right)$$

Then using partial fractions we get

$$A(x) = \frac{1}{1-x} + \frac{4x^2}{(1-2x)^2(1-x)} = \frac{5}{1-x} + \frac{-6}{1-2x} + \frac{2}{(1-2x)^2}.$$

The coefficient of $x^n$ in $1/(1-x)$ is 1, the coefficient of $x^n$ in $1/(1-2x)$ is $2^n$ and the coefficient of $x^n$ in $1/(1-2x)^2$ is $(n+1)2^n$ (the last using $1/(1-z)^t = \sum_{s\geq 0}\binom{s+t-1}{t-1}z^s$ with $z = 2x$ and $t = 2$ and $s = n$). Hence the coefficient of $x^n$ is $a_n = (5)(1) + (-6)(2^n) + (2)(n+1)2^n = 5 - 4 \cdot 2^n + n2^{n+1}$.

3.1.6 Define $a \prec b$ if and only if $b - a \geq 2$. Each $k$ subset of $[n]$ with no consecutive integers corresponds to a sequence $1 \leq a_1 \prec a_2 \prec \cdots \prec a_k \leq n$. Let $g_1 = a_1$, $g_{k+1} = n - a_k$ and for $i = 2, 3, \ldots, k$ let $g_i = a_i - a_{i-1}$. The properties of the $a_j$ imply that the $g_i$ are non-negative integers and in addition $g_1 \geq 1$ and $g_2, g_3, \ldots, g_{k-1}$ are each at least 2. Note also that $\sum_{i=1}^{k+1} g_i = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_k - a_{k-1}) + (n - a_k) = n$. So by looking at these gaps and recalling what we did for multisets we could directly determine $b_{n,k}$ (which we won't do here) and we can also use generating functions, getting the function as we did for multisets, with terms of the form $x^2 + x^3 + \cdots$ corresponding to $g_i \geq 2$ and then use the expansion for multisets to evaluate as follows:

$$\sum_{n\geq 0} b_{n,k}x^n = (x + x^2 + \cdots)(x^2 + x^3 + \cdots)^{k-1}(1 + x + x^2 + \cdots) = \frac{x^{2k-1}}{(1-x)^{k+1}}$$

and

$$\frac{x^{2k-1}}{(1-x)^{k+1}} = x^{2k-1}\sum_{m=0}^{\infty}\binom{m + (k+1) - 1}{(k+1) - 1}x^m = x^{2k-1}\sum_{m=0}^{\infty}\binom{m+k}{k}x^m.$$

Then $b_{n,k}$ is the coefficient of $x^n$ which is the coefficient of $x^{n-2k+1}$ in the last sum which is $\binom{(n-2k+1)+k}{k} = \binom{n-k+1}{k}$.

3.3.9 See also example 3.3.8 in the text for a different presentation of the solutions to the questions involving $a_{n,k}$.

(a) In the solution to 1.1.33 on the previous homework we showed $a_{n,k} = k!\binom{n+k-1}{k}$. In a similar manner to the version 1 proof we see that placing the unlabelled flags corresponds to positive integral solutions to $\sum_{i=1}^n x_i = k$ with the $x_i$ indicating how many flags are on pole $i$ (and hence positive in this case). This is $\binom{k-1}{n-1}$ and for each such solution we can label the flags in $k!$ ways so we get $b_{n,k} = \binom{k-1}{n-1}k!$.

(b) The exponential generating function for number of ways to place the flags on a single flagpole is $(0! + 1!\frac{x}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \cdots) = (1 + x + x^2 + x^3 + \cdots) = 1/(1-x)$ and if at least one flag must be on the pole the initial $0! = 1$ term is omitted and we get $(x+x^2+x^3+\cdots) = x/(1-x)$. Thus for $n$ poles we get

$$A(x) = \sum_{k\geq 0} \frac{a_{n,k}}{k!}x^k = \left(\frac{1}{1-x}\right)^n$$

and

$$B(x) = \sum_{k\geq 0} \frac{b_{n,k}}{k!}x^k = \left(\frac{x}{1-x}\right)^n.$$

(c) From (b) we see that $a_{n,k}/k!$ is the coefficient of $x^k$ in $(1/(1-x))^n$ which is $\binom{n+k-1}{k}$ and hence $a_{n,k} = k!\binom{n+k-1}{k}$. Also from (b) we see that $b_{n,k}/k!$ is the coefficient of $x^k$ in $(x/(1-x))^n$ which is the coefficient of $x^{k-n}$ in $(1/(1-x))^n$ which is $\binom{n+(k-n)-1}{(k-n)} = \binom{k-1}{(k-n)}$ and hence $b_{n,k} = k!\binom{k-1}{k-n} = k!\binom{k-1}{n-1}$.

3.4.9 (a) The Ferrers diagram for a partition of $2r+k$ into $r+k$ parts has $r+k$ rows so $r+k$ of the dots are in the first column and at most $r$ dots are not in the first column. Delete the first column to obtain a Ferrers diagram for an arbitrary partition of $r$. Conversely, given a Ferrers diagram for an arbitrary partition of $r$, add a first column of $r+k$ rows to obtain a Ferrers diagram for a partition of $2r+k$ into $r+k$ parts. Thus $p_{2r+k,r+k}$ is equal to the number of partitions of $r$, independent of $k$.

(b) Given a partition of $r+k$ into $k$ parts delete the first column of its Ferrers diagram to get a diagram for a partition of $r$ with at most $k$ rows. Taking the conjugate yields a diagram for a partition of $r$ with at most $k$ columns. It is straightforward to see that this can be reversed so $p_{r+k,k}$ counts the partitions of $r$ into parts of size at most $k$.

(c) Use the notation $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ with $1 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_k$ and $\sum \gamma_i = n$ for a partition of $n$ into $k$ parts (with the $\gamma_i$ indicating the sizes of the parts). Let $\gamma_i' = \gamma_i + i - 1$. Then the $\gamma_i'$ are distinct and $\sum \gamma_i' = \sum \gamma_i + (1 + 2 + \cdots (k-1)) = n + (k-1)k/2$. Hence the $\gamma'$ correspond to a partition of $n + (k-1)k/2$ into distinct parts. This can be reversed for $n \ge k$, given a partition of $n + (k-1)k/2$ into distinct parts (represented by $\gamma_i'$ such that $1 \le \gamma_1' < \gamma_2' < \cdots \gamma_k'$ with $\sum \gamma_i' = n + (k-1)k/2$) letting $\gamma_i = \gamma_i' - i + 1$ yields the $\gamma_i$ for a partition of $n$ into $k$ parts. Hence for $n \ge k$, $p_{n,k}$ counts the number of partitions of $n + (k-1)k/2$ into distinct parts. Letting $n = r + k$ yields the result in the problem.

(d) A partition of $n$ into $k$ parts either has a part of size one or it does not. If it does, deleting a 1 leaves a partition of $n-1$ into $k-1$ parts. If it does not, subtracting 1 from each part leaves a partition of $n-k$ into $k$ parts. It is straightforward to check that this can be reversed so that this is indeed a bijection. In terms of Ferrer's diagrams, we delete the last row if it has size 1 (leaving a diagram with 1 less dot and 1 less row) or delete the first column if the last row has at least 2 dots (leaving a diagram with $n-k$ dots and $k$ rows).

4.1.17 For the fall the number of assignments is the number of anagrams of the $2n$ symbols $1, 1, 2, 2, \ldots, n, n$. If $i$ appears in locations $r$ and $s$ then professor $i$ teaches courses $r$ and $s$. This number is $(2n)!/2^n$.

Let $A_i$ denote the event that professor $i$ teaches the same courses in the spring as in the fall. If $g_k$ counts the number of ways that a given set of $k$ professors teach the same course then this is the number of ways of assigning the remaining $n-k$ professors courses which from above is $(2n-2k)!/2^{n-k}$. Hence the number of events with none of the $A_i$ is $f(\emptyset) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g_k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-2k)!/2^{n-k}$. The probability is just this answer divided by the answer in the first paragraph.