Solutions to Homework 8
Combinatorics (Math 446) Fall 2004 Lehigh University

Due Monday November 1

37. This is clearly a hereditary system: if S has size at most k and is not a subset of any of the A_i then any $S' \subset S$ also has these properties. To see that it is non-empty we need an extra condition, that $|E| \geq k$. Then if $|A_i| < k$ for all i all size k subsets of E are bases. If $|A_i| \geq k$ for some i, choose $x \notin A_i$ (which exists as we assumed $A_i \neq E$) and $X \subseteq A_i$ with $|X| = k-1$. Then $X \cup \{x\}$ is a base. It has size k and if it was contained in any A_j then $|A_j \cap A_i| \geq k-2$.

Let B_1 and B_2 be two bases and $e \in B_1 - B_2$. Let $B_2 - B_1 = \{f_1, f_2, \ldots, f_t\}$. If $B - e + f_i$ is not a base then, since its size is k, for some $A_{\sigma(i)}$ we have $B_1 - e + f_i \subseteq A_{\sigma(i)}$. Assume that $B - e + f_i$ is not a base for all $f_i \in B_2 - B_1$. For f_i and f_j in $B_2 - B_1$ we get sets $A_{\sigma(i)}$ and $A_{\sigma(j)}$ with the size $k-1$ set $B_1 - e$ a subset of both. The intersection condition then implies $A_{\sigma(i)} = A_{\sigma(j)}$ (if they were distinct their intersection would be too large). So all of the $A_{\sigma(i)}$ are the same set, call it A. Now $B_2 \subseteq \bigcup_{i=1}^{t}(B_1 - e + f_i) \subseteq \bigcup_{i=1}^{t} A_{\sigma(i)} = A$ contradicting B_2 a base. Thus $B_1 - e + f_i$ is a base for some i and the base exchange axiom holds.

38. Let $\{e_1, e_2, \ldots, e_t\} = B_1 - B_2$ and $\{f_1, f_2, \ldots, f_t\} = B_2 - B_1$. Construct a bipartite graph H with parts $B_1 - B_2$ and $B_2 - B_1$. Put an edge in H between e_i and f_j if $B_1 - e_i + f_j$ is a base. A perfect matching in H gives the bijection. Use Hall's Theorem to show that H has a perfect matching. Consider $S \subseteq B_1 - B_2$. The B_2 is an independent set in $(B_1 \cup B_2) - S$. Thus we can augment the independent set $B_1 - S$ to an independent set $(B_1 - S) \cup T$ of size B_2 in $(B_1 \cup B_2) - S$. Note that $T \subseteq B_2 - B_1$ and has size $|S|$. Consider $f_j \in T$. $B_1 + f_j$ contains a circuit and this circuit must intersect S as otherwise it is a subset of the independent set $(B_1 - S) \cup T$. So for some $e_i \in B_1 - B_2$ we have $B_1 - e_i + f_j$ a base. This implies that T is contained in the neighborhood of S in H. So Hall's condition holds in H and we can find a perfect matching.

39. 18.1.23(a) By monotonicity of the rank function $r(X) \leq r(X + e + f)$ so we need to show $r(X) \geq r(X + e + f)$ from $r(X + e) = r(X + f)$ and submodularity. By submodularity $r((X + e) \cap (X + f)) + r((X + e) \cup (X + f)) \leq r(X + e) + r(X + f)$ which implies $r(X) + r(X + e + f) \leq r(X) + r(X)$ and cancelling gives $r(X+e+f) \leq r(X)$ as needed.

18.1.23(d) Assume that $x \in C_1 \cap C_2$. If $(C_1 \cup C_2) - x$ does not contain a circuit and is thus independent. By uniqueness $((C_1 \cup C_2) - x) + x = C_1 \cup C_2$ contains at most one circuit, a contradiction.

40. (18.2.2(a)): Note that $\overline{S}$ is independent in both dual matroids M_1^* and M_2^*. By the intersection formula there is some set Y attaining the minimum $|\overline{S}| = r_1^*(Y) + r_2^*(\overline{Y})$. Putting $\overline{Y}$ into the intersection formula for the original matroids M_1 and M_2 we get $|I| \leq r_1(\overline{Y}) + r_2(Y)$. With $|S| = |E| - |\overline{S}|$ and $|E| = |Y| + |\overline{Y}|$ we get

$$\begin{aligned} |I| + |S| &\leq r_1(\overline{Y}) + r_2(Y) + |E| - (r_1^*(Y) + r_2^*(\overline{Y})) \\ &= \big(|Y| - r_1^*(Y) + r_1(\overline{Y})\big) + \big(|\overline{Y}| - r_2^*(\overline{Y}) + r_2(Y)\big) \\ &= r_1(E) + r_2(E) \end{aligned}$$

where the last equality follows from the dual rank formula.

Similarly let X attain the minim in the intersection formula to get $|I| = r_1(X) + r_2(\overline{X})$ and in the duals we have $|\overline{S}| \leq r_1^*(\overline{X}) + r_2^*(X)$. With $|S| = |E| - |\overline{S}|$ this gives a lower bound on $|S|$. Then

$$\begin{aligned} |I| + |S| &\geq r_1(X) + r_2(\overline{X}) + |E| - (r_1^*(\overline{X}) + r_2^*(X)) \\ &= \big(|X| - r_2^*(X) + r_2(\overline{X})\big) + \big(|\overline{X}| - r_1^*(\overline{X}) + r_1(X)\big) \\ &= r_2(E) + r_1(E) \end{aligned}$$

where the last equality follows from the dual rank formula. Combining we get $|I| + |S| = r_1(E) + r_2(E)$.

(b) Let G be a bipartite graph with bipartition U_1, U_2 and let M_{U_i} be the partition matroid with a set of edge independent if and only if the endpoints in U_i are distinct. Thus the ranks of the matroids are $|U_i|$. A set of edges is independent in both M_{U_1} and M_{U_2} if and only if the edges have distinct ends in both U_1 and U_2. That is, if they form a matching. A set of edges is spanning in U_i if every vertex in U_i is the end of at least one of the edges (if not then edges incident to such a vertex are not in the span as adding them increases the rank). Thus a set of edges spanning both matroids is an edge cover of the vertices. For both of these the correspondence goes both ways so we have $\alpha'(G) = |I|$ and $\beta'(G) = |S|$. Then from part (a) we get $\alpha'(G) + \beta'(G) = |I| + |S| = r_{U_1}(E) + r_{U_2}(E) = |U_1| + |U_2| = n(G)$. Note that the set of vertices not covered by a set of edges independent in both matroids must form an independent set as otherwise we could add an edge. Thus $\alpha(G) \geq n(G) - \alpha'(G) = n(G) - (n(G) - \beta'(G)) = \beta'(G)$. Since independent vertices must be covered by distinct edges we also have $\alpha(G) \leq \beta'(G)$. Thus $\alpha(G) = \beta'(G)$.

41. Do 18.2.3: Let k denote the number of paths in a minimum disjoint path partition, n the number of vertices in G and use α for $\alpha(G)$ and β for $\beta(G)$. A set of edges is independent in both the head and tail partition matroids M_H and M_T if and only if all indegrees and outdegrees are at most one. That is, if and only if the edges form disjoint paths. In any forest with t components (possibly including some isolated

vertices, which will correspond to trivial paths in the path partition) the number of edges is $n - t$. Thus $k = n - |I|$ where I is a maximum size set independent in both matroids. Recall also Gallai's identity $\alpha+\beta = n$. Then using the matroid intersection formula $k = n - |I| = \alpha + \beta - |I| = \alpha + \beta - \min_{X\subseteq E}\left\{r_H(X) + r_T(\overline{X})\right\}$. From this $k \leq \alpha$ will follow if we show $\beta \leq \min_{X\subseteq E}\left\{r_H(X) + r_T(\overline{X})\right\}$. An independent set in the head partition matroid corresponds to a set of edges that induce a graph where every indegree is at most 1. That is, an inforest. Similarly, an independent set in the tail partition matroid corresponds to an outforest. For a given set of edges X let S be a maximal independent set in X. We have $r_H(X) = |S|$. Let R_H be the set of vertices with indegree 1 in the graph induced by S. Note that $|R_H| = |S| = r_H(X)$. These vertices cover the edges of X. Adding an edge not covered by R_H to S would increase the indegree of a vertex with indegree 0 and hence we would still have an independent set, contradicting maximality of S. In a similar manner we get a set R_T of $r_T(\overline{X})$ vertices covering the edges of $\overline{X}$ by looking at the tail partition matroid. Then $R_H \cup R_T$ is a vertex cover of the edges and $\beta \leq |R_H \cup R_T| \leq |R_H| + |R_T| = r_H(X) + r_T(\overline{X})$ and the result follows as this holds for all X.

42. Given a rainbow spanning tree and a partition $V_1, V_2, \ldots, V_k$ there are at least $k-1$ edges joint the parts as the tree is spanning. Since the edges must have different colors we have at least $k-1$ colors. It remains to show that if this color condition holds there is a rainbow spanning tree.

Let M_1 be the cycle matroid on the edges of G and let M_2 be the partition matroid on the edges with a set independent if and only if the edges have different colors. Then a set that is independent in both matroids is a rainbow forest. A common independent set of size $|V| - 1$ would be a rainbow spanning tree. Thus the result follows from matroid intersection if $\min_{X\subseteq E}\{r_1(X) + r_2(\overline{X})\} \geq |V| - 1$ as this would imply a common independent set of size $|V| - 1$. We will show that $r_1(X) + r_2(\overline{X}) \geq |V| - 1$ for all X. The graph G' induced by a set X of edges yields a partition of the vertices $V_1, V_2, \ldots, V_k$ with two vertices in the same part if and only if they are in the same component of G'. Since each component has a spanning tree there is a forest with $|V| - k$ edges among the edges of X. That is, $r_1(X) \geq |V| - k$ (in fact it is equal). The set $\overline{X}$ contains the set of edges between the parts of the partition and thus by assumption contains edges of at least $k-1$ colors. Thus $r_2(\overline{X}) \geq k - 1$. So $r_1(X) + r_2(\overline{X}) \geq (|V| - k) + (k - 1) = |V| - 1$ as needed.