More hints on homework 8.

Homework 8
Combinatorics (Math 446) Fall 2004 Lehigh University

40. (18.2.2(a)) - Note that $\overline{S}$ is independent in both dual matroids. So by the intersection formula there is some set Y attaining the minimum $|\overline{S}| = r_1^*(Y) + r_2^*(\overline{Y})$. Putting $\overline{Y}$ into the intersection formula for the original matroids we get $|I| \leq r_1(\overline{Y}) + r_2(Y)$. With $|S| = |E| - |\overline{S}|$ and $|E| = |Y| + |\overline{Y}|$ these give an upper bound on $|I| + |S|$ in terms of the various ranks in the above formulas and $|Y|, |\overline{Y}|$. Finally recall the formula for dual ranks which can be written as $r(E) = |X| - r^*(X) + r(\overline{X})$. Using this in the above bound, things work out exactly to make the upper bound $r_1(E) + r_2(E)$. To get the same lower bound switch the roles of the matroids and duals by starting with a set attaining the bound in the intersection formula for $|I|$ and using this set in the bound given by the intersection formula for the duals.

41. (18.2.3) - Observe that the number of paths in a path partition equals $|V(G)| -$ the number of edges in these paths. Also $|V(G)| = \alpha(G) + \beta(G)$ as we have seen before. The edges in a minimum path partition correspond to a set that is independent in both the head and tail partition matroid. Thus we get an equation involving $\alpha(G), \beta(G)$, the number of paths in a minimum path partition and the maximum size of a common independent set in the head and tail partition matroids. Let X be a set of edges attaining the bound in the matroid intersection formula. Look at the graph restricted to these edges. The edges of an independent set in the head partition matroid induce an outforest (all vertices have indegree at most 1). The number of edges is the number of non-root vertices (i.e., the vertices with indegree 1). If a set of edges is maximal and R the corresponding set of non-root vertices then every edge is incident to at least one vertex in R (otherwise we could form a larger forest). So R covers the edges in X. In a similar manner we can cover the edges of $\overline{X}$ using the tail partition matroid. This gives a bound on $\beta(G)$ which used in the equation described above yields the result.