Solutions to Homework 5
Combinatorics (Math 446) Fall 2004 Lehigh University

22. (a) Here we give an ad hoc argument that the maximum is 21. There are 6 vertices so at most 3 edges in a maximum matching. The weight 10 edges form two triangles so we can take at most 2 in a matching. The remaining edges have weight 1 so the maximum is at most 21 and this can be attained by taking the two vertical edges on the ends and the middle horizontal edge.

(b) Letting $x_e = 1/2$ for all weight 10 edges and $x_e = 0$ gives a feasible solution with value 30 as is easily checked. Letting $y_v = 5$ for all vertices gives a dual feasible solution of value 30 as is easily checked.

(c) Let $\pi_W = 9$ for each of the sets of three vertices that induce a triangle of weight 10 edges and let $\pi_W = 0$ otherwise and let $y_v = 1/2$ for all vertices. It is easy to check that this is a dual feasible solution with value 21. (Note, a feasible integral solution of value 21 also exists, put $y_v = 1$ for vertices in one triangle and $\pi_W = 8$ for this set of three vertices and the other three vertices put $y + v = 0$ and $\pi_W = 10$.)

23. Use the following procedure: Repeat II(i) or II(ii) until there are no intersections then repeat I until there are no intersections or inclusions. We will show below that this terminates with a minimum capacity cover with no inclusions or intersections.

I: If $v_j \in W_i$ let $u \in W_i$ be distinct from v_j with $u \notin \mathcal{W}$. Such a u exists as $|W_i| \geq 3$ and if all vertices in W_i were also in $\mathcal{W}$ then deleting W_i would give a smaller cover. To get $\mathcal{W}'$ from $\mathcal{W}$ delete W_i and add u and $W_i - \{u, v_j\}$ (if it has size at least 3, otherwise do not add it).

II(i): If $W_i \cap W_j = T$ with $|T|$ odd. To get $\mathcal{W}'$ from $\mathcal{W}$ delete W_i and W_j and add $W_i \cap W_j$.

II(ii): If $W_i \cap W_j = T$ with $|T| \geq 2$ and even. Note that $W_i \neq W_j$ or else we could delete one to get a smaller cover. So $W_i - W_j$ and $W_j - W_i$ are non-empty as their intersection has even size and they have odd size. Take any $u \in W_i - W_j$. To get $\mathcal{W}'$ from $\mathcal{W}$ delete W_i and W_j and add u and $(W_i \cup W_j) - \{u\}$.

If a minimum cover $\mathcal{W}$ has $W_i \cap W_j \neq \emptyset$ i, j apply II(i) or II(ii) as appropriate and repeat until there are no intersections. Note that if W'' intersects $W_i \cup W_j$ or $W_i \cup W_j - \{u\}$ then it intersected either W_i or W_j or both. So this strictly decreases the number of intersections of W_i and this part will indeed terminate with no intersections.

Given a cover with no intersections repeat I until there are no inclusions $v_j \in W_i$. Since there are no intersections u is not in any W_i' (so no new inclusions are created) and the number of inclusions decreases as the inclusion $v_j \in W_i$ is eliminated. Thus this will terminate in a cover with no intersections or inclusions as needed.

Finally we check that each procedure is indeed a cover and does not increase the capacity of the odd set cover so that the result will be a minimum cover with no intersections or inclusions.

For I: Since the only deletion from $\mathcal{W}$ is W_i, all edges are still covered in $\mathcal{W}'$ except possibly for those with both ends in W_i. Edges with both ends in $W_i - \{u, v_j\}$ are covered by that set in $\mathcal{W}'$. All other edges with both ends in W_i have at least on end either u or v_j and thus these are covered by u and v_j in $\mathcal{W}'$. The change in capacity is $capacity(\mathcal{W}') - capcity(\mathcal{W}) = 1 + (|W_i - \{u, v_j\}| - 1)/2 - (|W_i| - 1)/2$ where the $1 + (|W_i - \{u, v_j\}| - 1)/2$ comes from the additions of u and $W_i - \{u, v_j\}$ (when this set has size 1 and is not added the term $(|W_i - \{u, v_j\}| - 1)/2$ is also 0) and the deletion of W_i. So the capacity is unchanged by I.

For II(i): Since the only deletions from $\mathcal{W}$ are W_i and W_j, all edges are still covered in $\mathcal{W}'$ except possibly for those with both ends in W_i or with both ends in W_j. Both of these types of edges are covered by $W_i \cup W_j$. The change in capacity is $capacity(\mathcal{W}') - capcity(\mathcal{W}) = (|W_i \cup W_j| - 1)/2 - (|W_i| - 1)/2 - (|W_j| - 1)/2 \leq (|W_i| + |W_j| - 1 - 1)/2 - (|W_i| - 1)/2 - (|W_j| - 1)/2 \leq= 0$. The first $\leq$ follows since the intersection is non-empty. So II(i) does not increase capacity.

For II(ii): Since the only deletions from $\mathcal{W}$ are W_i and W_j, all edges are still covered in $\mathcal{W}'$ except possibly for those with both ends in W_i or with both ends in W_j. Both of these types of edges are covered by $W_i \cup W_j$ unless one end is u in which case the edge is covered by u. The change in capacity is $capacity(\mathcal{W}') - capcity(\mathcal{W}) = 1 + (|W_i \cup W_j - \{u\}| - 1)/2 - (|W_i| - 1)/2 - (|W_j| - 1)/2 \leq 1 + (|W_i| + |W_j| - 2 - 1 - 1)/2 - (|W_i| - 1)/2 - (|W_j| - 1)/2 \leq= 0$. The first $\leq$ follows since the intersection has size at least 2. So II(ii) does not increase capacity.

24. We need to use the matching duality theorem to prove Tutte's theorem: G has a perfect matching if and only if $odd(G - S) \leq |S|$ for all $S \subseteq V$. The necessity of the condition is straightforward and has been covered in class and in the text. It remains to show that if $odd(G - S) \leq |S|$ for all $S \subseteq V$ then G has a perfect matching. We prove the contrapositive: if G does not have a perfect matching then $odd(G - S) > |S|$ for some S. If G does not have a perfect matching then by the matching duality theorem and the disjointness result of the previous exercise there is a disjoint odd set cover with capacity less than $|V|/2$ (the number of edges in a perfect matching). Take a minimum odd set cover $\{W_1, W_2, \ldots, W_k, v_1, v_2, \ldots, v_r\}$. Let S be the set of vertices in the cover and T the set of vertices not in S or one of the W_i. Then $|V| = |S| + |T| + \sum_{i=1}^{k} |W_i|$. Every edge that does not have an end in S has both ends in some odd set W_i in the cover. Thus the components of $G - S$ are the W_i and isolated vertices T, all of which are odd. So $odd(G - S) = k + |T|$. From the capacity bound we have $|V|/2 > capacity(\mathcal{W}) = |S| + \sum_{i=1}^{k} (|W_i| - 1)/2$.

Combining all of this we get

$$|S| + |T| + \sum_{i=1}^{k} |W_i| = |V| > 2|S| + 2 \left(\sum_{i=1}^{k} \frac{|W_i| - 1}{2} \right) = 2|S| + \sum_{i=1}^{k} |W_i| + \sum_{i=1}^{k} (-1)$$

Thus $|T| > |S| - k$ and so $|S| < |T| + k = odd(G - S)$.

25. Note that shrinking a cycle creates a new vertex which becomes a root (an even vertex) of the alternating forest. Furthermore, note that the shrunken cycle was an even distance from the original root, so even vertices stay even in the new forest. Thus at the end of the algorithm all odd vertices correspond to vertices of G (not shrunken vertices). When we shrink a cycle the new vertex corresponds to an odd cardinality set of vertices in the previous graph. Thus after repeated shrinking (a previously shrunken cycle may be a vertex on a cycle that is shrink at a later stage etc) the number of vertices corresponding to a shrunken cycle is odd.

At the conclusion of the algorithm there are odd vertices of the final forest $v_1, v_2, \ldots, v_r$, even vertices of the final forest $w_1, w_2, \ldots, w_s$ and vertices not in the final forest $u_1, u_2, \ldots, u_t$. The vertices not in the final forest form a perfect matching in G' and each of the odd vertices is matched. So G' has a maximum matching of size $t/2 + r$. Some of the u_i and w_j vertices may be shrunken vertices resulting from repeated shrinking. (The v_i are not as noted above.) Let W_j and U_i be the sets of corresponding vertices in G. If the vertex is not shrunken this set will have size 1. By repeated applications of the shrinking lemma we get that the original graph G has a matching of size $t/2 + r + \sum_{i=1}^{t} (|U_i| - 1)/2 + \sum_{j=1}^{s} (W_j| - 1)/2$.

Let $U' = \bigcup_{i=1}^{t} U_i$. Let $V' = \{v_1, v_2, \ldots, v_r\} \cup \{u\}$ where u is some vertex of U' (if U' is empty then do not include u). Let $W' = \{W_j || W_j| \geq 3\} \cup (U' - u)$ (if $U' - u$ has size 1 do not include it). Then the vertices V' along with the odd sets W' form a minimum odd set cover of G. All edges among the U_i are covered by u and $U' - u$. All edges adjacent to a v_i are covered by that vertex. There are no edges between any W_j and another $W_{j'}$ or a U_i as otherwise the corresponding edge in the shrunken graph would have caused a shrinking or a growing of the final alternating forest. All edges between vertices in the same W_j are covered by the odd set. Thus this is a cover. Since $(|W_j| - 1)/2 = 0$ for the sets W_j of size 1 not included in the cover it is straightforward to check that this cover has size $t/2 + r + \sum_{i=1}^{t} (|U_i| - 1)/2 + \sum_{j=1}^{s} (W_j| - 1)/2$ as needed.

26. In the matching duality formula $\alpha'(G) = \beta^*(G)$ it is easy to see that $\alpha'(G) \leq \beta^*(G)$ from weak duality. So it remains to show $\alpha'(G) \geq \beta^*(G)$. Let $S \subseteq V$ be such that $\alpha'(G) = (|V| - (t - |S|))/2$ where $t = odd(G - S)$. Such a set exists by the Tutte-Berge formula. Let $W_1, W_2, \ldots W_t$ be the odd components of $G - S$ and U the vertices

in even components. Let $\mathcal{W}$ be the odd set cover consisting of vertices S and some vertex $u \in U$ (if U is non-empty) and odd sets those sets among $W_1, W_2, \ldots, W_t$ and $U - u$ that have size at least 3. (Note if S is maximal with equality then U will be empty. This can be shown in a manner similar to that in one of the proofs of Tutte's theorem in class.) This is indeed a cover as there are no edges between any W_j and another $W_{j'}$ or any vertex of U. Noting that $(|W_j| - 1)/2 = 0$ for the sets W_j of size 1 not included in the cover and that $|V| = |S| + |U| + \sum_{j=1}^{t} |W_j|$ we get the capacity of the cover to be

$$|S| + |U|/2 + \sum_{j=1}^{t}(|W_j| - 1)/2 = |S|/2 + (|S| + |U| + \sum_{j=1}^{t} |W_j|)/2 - t/2 = (|S| + |V| - t)/2 = \alpha'(G).$$

Thus $\beta^*(G) \leq \alpha'(G)$ as needed.

27. For $\alpha'(G) \leq (\min_{S \subseteq V} |V| - (odd(G - S) - |S|))/2$ note that every matching matches at most $|S|$ edges into odd components of $G - S$ so at least $odd(G_S) - |S|$ have an unsaturated vertex. Let $d = \max_{S \subseteq V}\{odd(G - S) - |S|\}$. We have $d \geq 0$ from $S = \emptyset$. Let G' be the graph obtained from G by adding a complete graph K_d on a set U of d new vertices and making each new vertex adjacent to all of V. Since d has the same parity as $|V|$ the number of vertices in G' is even and as G' is connected and G' satisfies Tutte's condition for $S = \emptyset$. For S' non-empty and $U \not\subseteq S'$ there is only one component in $G' - S'$ (as vertices of U are adjacent to all vertices in G') and $odd(G' - S') \leq 1 \leq |S'|$. If $U \subseteq S'$ let $S = S' - U$. Then note that $odd(G' - S') = odd(G - S)$ as the remaining graph is the same in each case. Then $odd(G' - S') = odd(G - S) \leq d + |S| = |S'|$. So Tutte's condition holds in G' and G' has a perfect matching. Restricting this to G yields a matching in G with at most d unsaturated vertices. Hence $\alpha'(G) \geq (\min_{S \subseteq V} |V| - (odd(G - S) - |S|))/2$ and the result follows.