Solutions to Homework 4
Combinatorics (Math 446) Fall 2004 Lehigh University

19. Let A be the arc set and V the node set of the digraph. In matrix notation, given vectors $\boldsymbol{c}, \boldsymbol{l}, \boldsymbol{u} \in \mathbf{R}^{|A|}$ and M the node-arc incidence matrix of the digraph, find $\boldsymbol{f} \in \mathbf{R}^{|A|}$ satisfying $\min\{\boldsymbol{cf} | M\boldsymbol{f} = \mathbf{0}, \boldsymbol{l} \le \boldsymbol{f} \le \boldsymbol{u}\}$. Rewriting this in typical primal format we have $\max\left\{(-\boldsymbol{c})\boldsymbol{f} \,\middle|\, \begin{bmatrix} M \\ I \\ -I \end{bmatrix} \boldsymbol{f} \begin{matrix} = \\ \le \\ \le \end{matrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{u} \\ -\boldsymbol{l} \end{bmatrix}\right\}$. In Σ notation we have $\min \sum_{uv \in A} c(uv) f(uv)$ s.t. $\sum_{uv \in A} f(uv) - \sum_{vw \in A} f(vw) = 0$ for all $v \in V$ and $l(uv) \le f(uv) \le u(uv)$ for all $uv \in A$.

The dual (taken from the 'standard form') is
$\min\left\{\begin{bmatrix} \boldsymbol{\pi} & \boldsymbol{a} & \boldsymbol{b} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{u} \\ -\boldsymbol{l} \end{bmatrix} \,\middle|\, \begin{bmatrix} \boldsymbol{\pi} & \boldsymbol{a} & \boldsymbol{b} \end{bmatrix} \begin{bmatrix} M \\ I \\ -I \end{bmatrix} = -\boldsymbol{c}, \boldsymbol{a} \ge \mathbf{0}, \boldsymbol{b} \ge \mathbf{0}\right\}$. Which is $\max\{\boldsymbol{bl} - \boldsymbol{au} | \boldsymbol{b} - \boldsymbol{a} - \boldsymbol{\pi}M = \boldsymbol{c}, \boldsymbol{a} \ge \mathbf{0}, \boldsymbol{b} \ge \mathbf{0}\}$. In Σ notation we have $\max \sum_{uv \in A} (l(uv)b(uv) - u(uv)a(uv))$
s.t. $b(uv) - a(uv) + \pi(v) - \pi(u) = c(uv)$ for all $uv \in A, a(uv), b(uv) \ge 0$ for all $uv \in A$.

20. The directed Chinese Postman problem is solved by solving the min cost circulation problem on the given digraph where the costs on the arcs are the arc lengths, the lower bounds are all 1 and the upper bounds are infinite. (Note that we could use this model to solve the more general problem where only some of the arcs need to be traversed by putting a lower bound of 0 on the other arcs.) We use integrality from total unimodularity of the constraint matrix to ensure that our solution corresponds to a walk, the value of the flow variable giving the number of times an arc is traversed.

For the unweighted version we also have all of the costs equal to 1. Observe that in the dual with with infinite upper bounds the variables $a(uv)$ will all be 0 in a finite optimal solution. Thus the dual in this case becomes
$\max \sum_{uv \in A} b(uv)$ s.t. $b(uv) + \pi(v) - \pi(u) = 1$ for all $uv \in A$. Take an integral optimal dual solution $\boldsymbol{b}^*, \boldsymbol{\pi}^*$. The existence of an integral optimal solution follows from total unimodularity of the constraint matrix. The dual has a finite optimum because the primal does (this existence was shown in a previous homework). That the primal has a finite optimum follows from the assumption that the digraph is strongly connected.

Since the $\pi^*(v)$ are free variables, do not appear in the objective function and in each equation they occur in pairs, one positive and one negative, we can add a constant to all of them and obtain another feasible optimal solution. Thus we may assume

that the smallest $\pi^*(v)$ is 1. From $b^*(uv) + \pi^*(v) - \pi^*(u) = 1$ and $b^*(uv) \geq 0$ we get $\pi^*(v) \leq \pi^*(u) \leq 1$. Let $R_i = \{v \in V | \pi^*(v) \leq i\}$ and consider the cuts $[R_i, \overline{R_i}]$ and $[\overline{R_i}, R_i]$. From $\pi^*(v) \leq \pi^*(u) \leq 1$, we have that $[R_i, \overline{R_i}]$ contains only arcs uv with $\pi^*(u) = i$ and $\pi^*(v) = i+1$. So the cuts $[R_i, \overline{R_i}]$ are disjoint. If $\pi^*(v) = \pi^*(u) + 1$ then uv is only in the cut $[R_i, \overline{R_i}]$ (among those listed above). If $\pi^*(v) = r \leq s = \pi^*(u)$ then $b^*(uv) = 1 - r + s$ and uv is in the $s - r$ cuts $[\overline{R_r}, R_r], \ldots, [\overline{R_{s-1}}, R_{s-1}]$ (there are none if $r = s$). The objective value is as follows. The sums are assume to be over arcs uv plus any other noted conditions: $\sum b^*(uv) = \sum (1 - \pi^*(v) + \pi^*(u)) = \sum 1 + \sum\limits_{\pi^*(v) - \pi^*(u) = 1} (-1) + \sum\limits_{\pi^*(v) \leq \pi^*(u)} (\pi^*(u) - \pi^*(v))$.
The first term is just $|A|$. By the discussion above, the second term is the -1 times the sum of the $[R_i, \overline{R_i}]$ and the third term is the sum of the $[\overline{R_i}, R_i]$. Thus we get that the objective value is $|A| + \sum\limits_{R \in \mathcal{R}} \left(|[\overline{R}, R]| - |[R, \overline{R}]|\right)$. This shows *geq* in the max-min formula. As noted in class $\leq$ is straightforward to check.

21. A family attaining the bound is the $\binom{n-1}{k-1}$ size k subsets of $[n] = \{1, 2, \ldots, n\}$ containing 1. Let $\binom{[n]}{k}$ denote the collection of all size k subsets of $\{1, 2, \ldots, n\}$ and let Π'_n denote the set of all $(n-1)!$ cyclic permutations of $[n]$. For a given intersecting family $\mathcal{F}$ we can consider variables x_A where A ranges over $\binom{[n]}{k}$ with $x_A = 1$ if $A \in \mathcal{F}$ and 0 otherwise. The comments on cyclic permutations imply that $\sum\limits_{A \in Q(\sigma)} x_A \leq k$ for any cyclic permutation σ. Thus the integer linear programming problem $\max \left\{ \sum\limits_{A \in \binom{[n]}{k}} x_A \text{ s.t. } \sum\limits_{A \in Q(\sigma)} x_A \leq k \text{ for } \sigma \in \Pi_n, x_A \in \{0,1\} \text{ for } A \in \binom{[n]}{k} \right\}$.
Provides an upper bound on the maximum size of an intersecting family. Relaxing the constraint $X_A \in \{0,1\}$ to $x_A \geq$ also gives a bound (as the max in the relaxed problem is at least as large as the integer program). Each set A in is $Q(\sigma)$ for $k!(n-k)!$ different σ (place any permutation of the set in one part of the circle in $k!$ ways and any permutation of its complement in the rest in $(n-k)!$ ways). Thus, each variable x_A appears in exactly $k!(n-k)!$ of the inequalities, each time with coefficient 1. Let the $(n-1)!$ dual variables be y_σ. Then, since the right side of the dual is $\mathbf{1}$, setting all dual variables to $y^*_{sigma} = 1/(k!(n-k)!)$ is dual feasible (as each inequality in dual then becomes $1 \geq 1$). The value of this is $\sum\limits_{\sigma \in \Pi_n} k y^*_\sigma = \sum\limits_{\sigma \in \Pi_n} \dfrac{k}{k!(n-k)!} = (n-1)! \dfrac{k}{k!(n-k)!} = \dbinom{n-1}{k-1}$. This is an upper bound on the dual optimal which by weak duality is an upper bound on the primal optimal which by the comments above is an upper bound on the size of an intersecting family.