Homework 4 Combinatorics (Math 446) Fall 2004 Lehigh University

Due Wednesday September 22

19. Write down a linear program to solve the minimum cost circulation problem: given a digraph with lower bounds l, upper bounds u and costs c on the arcs (i.e., l(xy), u(xy) and c(xy) for each arc xy) with $l \leq u$ determine the minimum cost of a feasible circulation. Give the dual to the problem. For both the primal and dual give your answer in two forms, once using matrix notation and once using σ notation.

20. Recall that the directed Chinese postman problem is to find a minimum cost of a closed walk traversing each arc at least once. (You can assume that the digraph is what is called strongly connected: it is possible to get from any vertex to another by some directed walk. Otherwise there is no closed walk traversing each arc at least once.) Explain how this can be solved as a minimum cost circulation problem. For the unweighted version the costs on each arc are 1 and the problem is to find the minimum number of arcs in a closed walk traversing each arc at least once. For this problem use the duality result of problem 19 and total unimodularity to prove the following: The minimum number of arcs in a closed walk traversing each arc at least once is equal to the maximum $|A| + \sum_{R \in \mathcal{R}} \left(|[\overline{R}, R]| - |[R, \overline{R}]| \right)$ where A is the arc set

and \mathcal{R} is a family of subsets of the vertex set such that the cuts $[R, \overline{R}]$ are disjoint (i.e., for R, R' in the family $[R, \overline{R}] \cap [R', \overline{R'}] = \emptyset$). Hint, there is a dual variable associated with each vertex, consider sets R_i consisting of vertices whose dual variable value is at most *i* for some optimal dual solution.

21. A family A_1, A_2, \ldots, A_t of size k subsets of $\{1, 2, \ldots, n\}$ is intersecting if $A_i \cap A_j \neq \emptyset$ for all pairs i, j. The Erdos-Ko-Rado Theorem states that for $k \leq n/2$ the maximum size of an intersecting family of size k subsets of an n set is $\binom{n-1}{k-1}$. Prove this as follows. Describe a family attaining the bound. Show that $\binom{n-1}{k-1}$ is an upper bound on all such families as follows. Create a variable X_A for each size k subset A which we will take on values 0 or 1. We interpret 1 as being in the family and 0 as not being in the family. Let $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ be a cyclic permutation of $\{1, 2, \ldots, n\}$ and let $Q(\sigma)$ be the collection of subsets $Q_j = \{\sigma_j, \sigma_{j+1}, \ldots, \sigma_{j+k-1}\}$ for $j = 0, 1, \ldots, n-1$ with addition on the subscripts modulo n. It is not hard to show that at most k of the subsets in a $Q(\sigma)$ can be in an intersecting family. You can assume this. The system consisting of the inequality that this implies (for all cyclic permutations) gives an upper bound on the size of an intersecting family. Use a dual solution and weak duality to prove the upper bound $\binom{n-1}{k-1}$. Hint - try making all of the dual variables equal.