Solutions to combinatorics (math 446) homework 3, fall 2004, Lehigh University

13. Given G construct a network D with

$$V(D) = \{v^-, v^+ | v \in V(G)\}$$
 and $A(D) = \{v^+ w^- | vw \in A(G)\} \cup \{v^- v^+ | v \in V(G)\}$

For capacities, all arcs in D have lower bound 0. Arcs of the form v^+w^- have infinite upper bound and arcs of the form v^-v^+ have upper bound 1 except for the arcs x^-x^+ and y^-y^+ which have infinite upper bound. Let x^- be the source in D and y^+ be the sink. Since the only arc leaving a vertex v^- in D is of the form v^-v^+ there is a one to one correspondence between x - y paths in G and $x^- - y^+$ paths in D. the upper bound of 1 on arcs v^-v^+ ensures that each vertex pair other than x^-, x^+ and y^-y^+ is used at most once in a maximum flow in D. Then, since flows decompose into paths, a maximum flow in D corresponds to internally disjoint paths in G. Let [S,T] be cut with finite capacity m in D. It contains m arcs v^-v^+ for $v \in V(G) - \{x, y\}$ as all other arcs have infinite capacity. Let R be the set of vertices v in V(G) with $v^- \in S$ and $v^+ \in T$. By the path correspondence, every x - y path in G must go through a vertex of R. Thus R is a separating set of G. Then $|R| = m = \max$ flow = max number of internally disjoint x - y paths .

14. This can be done with just about any of the network methods we have used. We will use max-flow min-cut here. Construct a network D with $V(D) = \{s,t\} \cup X \cup Y \ A(D) = \{sx_i | x_i \in X\} \cup \{y_j t | y_j \in Y\} \cup \{x_i y_j | xy \in E(G)\}$. All lower bounds are 0. The upper bounds are given by $u(sx_i) = \sigma(i), u(y_j t) = \delta(j)$, and $u(x_i y_j) = \infty$. Given a flow in D let z_{ij} be the flow on the arc $x_i y_j$. The upper bound $\sigma(i)$ on the only arc entering x_i ensures that $\sum_{j \in N(i)} z_{ij} \leq \sigma(i)$

(as the term on the left is the sum of the flows on the arcs leaving x_i) for all $i \in X$. Similarly, $\sum_{i \in N(j)} z_{ij} \leq \delta(j)$ for all $j \in Y$. A flow of value $\sum_{y_j \in Y} \delta(j)$ will force $\sum_{i \in N(j)} z_{ij} = \delta(j)$. Thus we

need to show that there is a flow of this value if and only if the given condition holds.

Consider a minimum cut [S,T]. Let $U = T \cap Y$. Since the cut is finite, there are no arcs $x_i y_j$ with $x_i \in S$ and $y_j \in T$. Thus, in G there are no corresponding edges xy and $N(U) \subseteq T \cap X$. If $x_i \in T \cap X$ is not in N(U) then moving x_i to S would would not add any infinite upper bound arcs to the cut and would decrease the capacity by $\sigma(i)$. Thus, for a minimum cut with $U = T \cap Y$ we have $N(U) = T \cap X$. The capacity of such a minimum cut is $\sum_{x_i \in T \cap X} \sigma(i) + \sum_{y_i \in S \cap Y} \delta(j)$ and

$$\sum_{x_i \in T \cap X} \sigma(i) + \sum_{y_j \in S \cap Y} \delta(j) \ge \sum_{y_j \in Y} \delta(j) \Leftrightarrow \sum_{x_i \in T \cap X} \sigma(i) \ge \sum_{y_j \in Y \cap T} \delta(j)$$

With $U = T \cap Y$ and $N(U) = T \cap X$ this is exactly the condition.

15. Let \mathcal{F} be an intersecting family. Consider variables x_{σ} for each k permutation with $x_{\sigma} = 1$ if σ is in \mathcal{F} and $x_{\sigma} = 0$ otherwise. Call two k permutations $\langle \sigma(1)\sigma(2)\cdots\sigma(k) \rangle$ and $\langle \gamma(1)\gamma(2)\cdots\gamma(k) \rangle$ equivalent if there is some constant c such that $\sigma(i) - \gamma(i) = c$ for $i = 1, 2, \ldots, k$ where addition is modulo n (and we write n for 0). This is clearly an equivalence relation. The equivalence class containing a given σ is the set of all permutations equivalent to σ . (For example, with n = 7 and k = 4 one equivalence class is $\{1537, 2641, 3752, 4163, 5274, 6315, 7426\}$.) The set \mathcal{C} of equivalence class partitions the set of permutations. Note that no two permutations in an equivalence class intersect. Thus $\sum_{\sigma \in C} x_{\sigma} \leq 1$ for each equivalence class. Then

$$\sum_{\sigma \in \mathcal{F}} x_{\sigma} = \sum_{C \in \mathcal{C}} \sum_{\sigma \in C} x_{\sigma} \le \sum_{C \in \mathcal{C}} 1 = \frac{(n-1)!}{(n-k)!}$$

The first equality follows since the equivalence classes partition the set of k permutations, the second was observed above and the last follows as the n!/(n-k)! permutations are partitioned into equivalence classes of size n so there are (n-1)!/(n-k)! equivalence classes.

Thus we have the correct upper bound on the size of \mathcal{F} . (The underlying idea is weak duality.) To show that the bound is attained consider the family of all permutations with first entry 1. These clearly intersect and there are (n-1)!/(n-k)! of them.

16. For simplicity in notation let ST denote the set of arcs from S to T. Let u(ST) denote the sum of the upper bounds on the arcs in ST and similarly for l. Also, let $A = S \cap T$, $B = S \cap \overline{T}$, $C = \overline{S} \cap T$ and $D = \overline{S} \cap \overline{T}$. Then

$$cap(S \cup T, \overline{S \cap T}) = u(AD) + u(BD) + u(CD) - l(DA) - l(DB) - l(DC)$$

$$cap(S \cap T, \overline{S \cap T}) = u(AB) + u(AC) + u(AD) - l(BA) - l(CA) - l(DA)$$

and

$$cap(S,\overline{S}) = u(AC) + u(AD) + u(BC) + u(BD) - l(CA) - l(DA) - l(CB) - l(DB) cap(T,\overline{T}) = u(AB) + u(AD) + u(CB) + u(CD) - l(BA) - l(DA) - l(BC) - l(DC)$$

It is straightforward to check that the sum of the second two right sides minus the sum of the first two is u(BC) + u(CB) - l(CB) - l(BC). Since we assume that upper bounds are at least the lower bounds for each arc we have $u(BC) - l(BC) \ge 0$ and $u(CB) - l(CB) \ge 0$ and hence the sum of the second two right sides minus the sum of the first two is at least 0. This is the inequality to be shown.

17. We may assume that $s_1 \leq s_2 \leq \cdots \leq s_n$. (Note then that since the r_i and s_i are paired then we cannot assume anything about the r_i , however this will not be necessary). By the Gale-Ryser Theorem there is a bipartite graph with degrees (r_1, r_2, \ldots, r_n) and (s_1, s_2, \ldots, s_n) if and only if $\sum i = 1^m \min\{r_i, k\} \geq \sum_{j=1}^k s_j$ for $k = 1, 2, \ldots n$. (Note that the condition Gale-Ryser does not need monotonicity for the p_i .) Finally, observe that there is a bipartite graph with these degrees if and only if there is a digraph with these as indegrees and outdegrees. Let the vertex set of the digraph be $\{v_1, v_2, \ldots, v_n\}$ and the parts of the bipartite graph $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. Put arc $v_i v_j$ in the digraph if and only if $x_i y_j$ is an edge of the bipartite graph. 18. Let (I) be the statement $\max{\{cx|Ax = b, x \ge 0\}} = \min{\{yb|yA \ge c\}}$ (when both are feasible) and (II) the statement $\max{\{cx|Ax \le b\}} = \min{\{yb|yA = c, y \ge 0\}}$ (when both are feasible).

To show (II) implies (I): Assuming the first and last LPs below are feasible we have

$$\max\{c\boldsymbol{x}|A\boldsymbol{x}=\boldsymbol{b},\boldsymbol{x}\geq\boldsymbol{0}\} = \max\left\{\left[\frac{A}{-A}\right]\boldsymbol{x}\leq\left[\frac{\boldsymbol{b}}{-\boldsymbol{b}}\right]\right\}$$
$$= \min\left\{\left[\boldsymbol{u}\mid\boldsymbol{v}\mid\boldsymbol{w}\right]\left[\frac{\boldsymbol{b}}{-\boldsymbol{b}}\right]\mid\left[\boldsymbol{u}\mid\boldsymbol{v}\mid\boldsymbol{w}\right]\left[\frac{A}{-A}\right]=\boldsymbol{c},\left[\boldsymbol{u}\mid\boldsymbol{v}\mid\boldsymbol{w}\right]\geq\boldsymbol{0}\right\}$$
$$= \min\left\{\boldsymbol{y}\boldsymbol{b}|\boldsymbol{y}A=\boldsymbol{c},\boldsymbol{y}\geq\boldsymbol{0}\right\}$$

The first and third equalities follow from basic manipulations. The second follows from (II). To show (I) implies (II): Assuming the first and last LPs below are feasible we have

$$\max \{ \boldsymbol{c} \boldsymbol{x} | A \boldsymbol{x} \le \boldsymbol{b} \} = \max \left\{ \begin{bmatrix} \boldsymbol{c} \mid -\boldsymbol{c} \mid \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \hline -\boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} \mid \begin{bmatrix} A \mid -A \mid I \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \hline -\boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = \boldsymbol{b}, \begin{bmatrix} \boldsymbol{u} \\ \hline -\boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} \ge \boldsymbol{0} \right\}$$
$$= \min \left\{ \boldsymbol{y} \boldsymbol{b} | \boldsymbol{y} \begin{bmatrix} A \mid -A \mid I \end{bmatrix} \ge \begin{bmatrix} \boldsymbol{c} \mid -\boldsymbol{c} \mid \boldsymbol{0} \end{bmatrix} \right\}$$
$$= \min \{ \boldsymbol{y} \boldsymbol{b} | \boldsymbol{y} A = \boldsymbol{c}, \boldsymbol{y} \ge \boldsymbol{0} \}$$

The first and third equalities follow from basic manipulations. The second follows from (I).