Homework 3
Combinatorics (Math 446) Fall 2004 Lehigh University

Due Wednesday September 15

13. There are several versions of Menger's Theorem (see pages 340-343 in the text). Consider the following: In a digraph the maximum number of internally disjoint paths from x to y is equal to the minimum size of an x, y separating set. (A set of paths from x to y is internally disjoint if the no two paths share a vertex other than x and y. An x, y separating set is a set of vertices not including x or y whose deletion leave no paths from x to y.) Prove this by using the max-flow min-cut theorem.

14. Assume that G is a bipartite graph with bipartition X, Y and supplies $\sigma(i)$ for $i \in X$ and demands $\delta(j)$ for $j \in Y$. Let $N(i)$ denote the set of neighbors of vertex i. That is, $N(i) = \{j | ij \in E(G)\}$ where $E(G)$ is the edge set of G. Similarly, the neighborhood of a set U is the set of vertices adjacent to some vertex in U. The demand is satisfied if there exists z_{ij} defined on pairs i, j with ij and edge of G such that $\sum_{i \in N(j)} z_{ij} = \delta(j)$ for all $j \in Y$ and $\sum_{j \in N(i)} z_{ij} \leq \sigma(i)$ for all $i \in X$. Prove (in any manner you choose) that the demands can be satisfied if and only if for all $U \subseteq Y$ we have $\sum_{j \in U} \delta(j) \leq \sum_{i \in N(U)} \sigma(i)$. (Note - this is related to but not the same as example 19.2.24 and it is a generalization of Hall's Theorem on page 295.)

15. Let σ and π be permutations of $k-$element subsets of n. Say that σ and π intersect if $\sigma(i) = \pi(i)$ for some i. Prove that the maximum size of a pairwise intersecting family of such $k-$permutations of $[n] = \{1, 2, \ldots, n\}$ is $(n-1)!/(n-k)!$. Hint - it is easy to construct a family meeting the bound. To show there is no larger family use what can be viewed as an elementary application of weak duality. Take a permutation along with all permutations obtained by cyclically shifting it. Consider how many of these can be in an intersecting family and obtain an inequality. Combine these inequalities to obtain the result.

16. Do 19.2.8 in the text

17. Do 19.2.11 in the text

18. Prove the equivalence of the strong duality theorems for the following primal-dual pairs: $\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}\}$ and
$\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$