7. Weak duality shows that if the primal is feasible then the dual is either bounded or infeasible as $\boldsymbol{cx}$ for any primal feasible $\boldsymbol{x}$ gives a lower bound on the dual. Assume that the dual is infeasible. Then there is no solution to $(-A^T)\boldsymbol{y}^T \leq -\boldsymbol{c}^T, \boldsymbol{y} \geq \boldsymbol{0}$ and by Farkas' lemma there is a solution to $\boldsymbol{w}(-A^T) \geq \boldsymbol{0}, \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{w}(-\boldsymbol{c})^T < 0$. That is, $A\boldsymbol{w}^T \leq \boldsymbol{0}, \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{cw}^T > 0$. Let $\boldsymbol{w}^*$ be such a solution and let $\boldsymbol{x}^*$ be an optimal primal solution. Then, for any positive M, we have $\boldsymbol{x}^* + M\boldsymbol{w}^{*T}$ primal feasible as $\boldsymbol{x}^* \geq \boldsymbol{0}$ and $\boldsymbol{w}^* \geq \boldsymbol{0}$ and $A(\boldsymbol{x}^* + M\boldsymbol{w}^{*T}) \leq \boldsymbol{b} + M\boldsymbol{0} = \boldsymbol{b}$. Also, $\mathbf{c}(\boldsymbol{x}^* + M\boldsymbol{w}^{*T}) > \boldsymbol{cx}^*$ since $\boldsymbol{cw}^{*T} > 0$. This contradicts the optimality of $\boldsymbol{x}^*$. So the dual must be feasible and bounded.

8. Let f^* be a circulation with the minimum number (among all circulations) of arcs not at a bound. That is, f^* minimizes the number of arcs with $l(xy) < f^*(xy) < u(xy)$. The result will be proved by assuming that this number is at least n and reaching a contradiction. If the number of such arcs is at least n then consider the digraph induced by the set of arcs not at a bound. Ignoring directions on the arcs we have a graph with at least n edges on n vertices. Such a graph must contain a cycle. A cycle in this graph is of the form $x_0x_1\ldots x_t$ where either x_ix_{i+1} or $x_{i+1}x_i$ is an arc for $i = 1, 2, \ldots, t$ (subscript addition modulo $t+1$). Call an arc forward if it is of the form x_ix_{i+1} and backward if it is $x_{i+1}x_i$. Let $\gamma_f = \min\{u(x_ix_{i+1}) - f^*(x_ix_{i+1}) | x_ix_{i+1} \text{ is forward }\}$ and $\gamma_b = \min\{f^*(x_{i+1}x_i) - l(x_{i+1}x_i) | x_{i+1}x_i \text{ is backward }\}$ and let $\gamma = \min\{\gamma_f, \gamma_b\}$. Form a now flow g by letting $g(x_ix_{i+1}) = f^*(x_ix_{i+1}) + \gamma$ for forward arcs, $g(x_{i+1}x_i) = f^*(x_{i+1}x_i) - \gamma$ for backward arcs and $g(xy) = f^*(xy)$ for all other arcs. It is straightforward to check that g is a circulation, arcs that were at a bound in f^* are also in g and that arcs that attain the minimum for γ are also now at a bound. This contradicts the choice of f^* as minimizing the number of arcs at a bound. Thus the number of arcs not at a bound must be at most $n-1$.

9. As in the proof of the max-flow min-cut theorem $max \leq min$ is easy to show. So we need to show that $max \geq$ min. Consider a network N with with a feasible flow and a minimum cut size ν. Add an arc ts with lower bound ν and upper bound ∞ to form a digraph D. If we can show that there is a circulation in D then there is an $s-t$ flow of value at least ν in N, establishing $max \geq min$. If there is no circulation, then there is a set of nodes T with $\sum_{x \notin T, y \in T, xy \in A(D)} u(xy) < \sum_{y \in T, z \notin T, xy \in A(D)} l(yz)$. Since the upper bound on ts is infinite we cannot have $s \in T$ and $t \notin T$. If $s, t \in T$ then T would also establish that there is no circulation in the digraph formed by putting lower and upper bounds on ts equal to the value of some feasible $s-t$ flow. This would contradict the feasibility of the flow. Thus we can assume that $s \notin T$ and $t \in T$. Then, separating out the bound on arc ts in the term on the right we have $\sum_{x \notin T, y \in T, xy \in A(N)} u(xy) < \nu + \sum_{y \in T, z \notin T, xy \in A(N)} l(yz)$. Thus, in N the cut $[\overline{T}, T]$ has capacity $\sum_{x \notin T, y \in T, xy \in A(D)} u(xy) - \sum_{y \in T, z \notin T, xy \in A(D)} l(yz) < \nu$, contradicting the minimality of ν.

10. Any solution to $A'\boldsymbol{x}' \leq \boldsymbol{b}', \boldsymbol{x}' \geq \boldsymbol{0}$ solves $A'\boldsymbol{x}' - \boldsymbol{b}'z \leq \boldsymbol{0}, \boldsymbol{x}' \geq \boldsymbol{0}, z > 0$ by taking $z = 1$. Conversely, assume that $A'\boldsymbol{w}' - \boldsymbol{b}'z \leq \boldsymbol{0}, \boldsymbol{w}' \geq \boldsymbol{0}, z > 0$. Divide the first inequality by $z > 0$ and rearrange terms to get $A'\boldsymbol{w}'/z \leq \boldsymbol{b}'$. Since $z > 0$ we have $\boldsymbol{w}'/z \geq \boldsymbol{0}$. Thus $\boldsymbol{x}' = \boldsymbol{w}'/z$ solves $A'\boldsymbol{x}' \leq \boldsymbol{b}', \boldsymbol{x}' \geq \boldsymbol{0}$.

11. A feasible solution to (2) in the notes on proving duality from the theorem of the alternative yileds both primal and dual optimal solutions. Homogenizing (2) as in the hint we get

$$\left[\begin{array}{c|c|c} A & 0 & -\boldsymbol{b} \\ \hline -\boldsymbol{c} & \boldsymbol{b}^T & 0 \\ \hline 0 & -A^T & \boldsymbol{c}^T \end{array}\right] \leq \mathbf{0}, \left[\begin{array}{c} \boldsymbol{x} \\ \hline \boldsymbol{y}^T \\ \hline z \end{array}\right] \geq \mathbf{0} \text{ with } z > 0$$

Permuting the row order this becomes

$$\left[\begin{array}{c|c|c} 0 & -A^T & \boldsymbol{c}^T \\ \hline A & 0 & -\boldsymbol{b} \\ \hline -\boldsymbol{c} & \boldsymbol{b}^T & 0 \end{array}\right] \leq \mathbf{0}, \left[\begin{array}{c} \boldsymbol{x} \\ \hline \boldsymbol{y}^T \\ \hline z \end{array}\right] \geq \mathbf{0} \text{ with } z > 0 \tag{1}$$

Consider a game with payoff matrix P equal to the matrix on the left in (1). Since P is skew-symmetric, the game is symmetric and has value 0. Thus, player 2 has a strategy $\boldsymbol{q}$ such that $P\boldsymbol{q} \leq \mathbf{0}$. (That is, a strategy such that for each row player 1 can expect to 'win' at most 0.) This strategy gives a solution to the homogenized system above if as we noted we have $z > 0$. Then partitioning $\boldsymbol{q}^T = \left[\begin{array}{c|c|c} \boldsymbol{x}^T & \boldsymbol{y} & z \end{array}\right]$ we have $\boldsymbol{x}/z$ primal optimal and $\boldsymbol{y}/z$ dual optimal. (These solve the system (2) as in the proof of problem 10 and a solution to (2) gives primal and dual optimal solutions as in the notes.)

For the given matrix we get the partitioned matrix

$$\left[\begin{array}{rrr|rr|r} 0 & 0 & 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & -1 & -4 & 9 \\ 0 & 0 & 0 & -2 & -5 & 0 \\ \hline 0 & 1 & 2 & 0 & 0 & -6 \\ 3 & 4 & 5 & 0 & 0 & -7 \\ \hline -8 & -9 & 0 & 6 & 7 & 0 \end{array}\right]$$

12. (a) Construct a network with nodes $\{s,t\} \cup \{x_1, x_2, \ldots, x_p\} \cup \{y_1, y_2, \ldots, y_q\}$ and for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$ arcs $x_i y_j$ with upper bound 1, sx_i with upper bound m_i and $y_j t$ with upper bound n_j. All lower bounds are 0. By the integrality theorem we can assume that there is a maximum flow with the flow on all arcs $x_i y_j$ equal to 0 or 1. A flow of 1 indicates a worker from firm i going to group j. The upper bounds n_j on arcs $y_j t$ allow at most n_j workers in group j. A flow of value $\sum_{i=1}^{p} m_i$ saturates all arcs sx_i and thus for each i has m_i arcs $x_i x_j$ with flow 1, allowing all workers to be assigned to groups.

(b) Consider a cut $[S,T]$ in the network. If $y_{j'} \in S$ and $y_j \notin S$ for $j < j'$, the cut $[S - y_{j'} \cup \{y_j\}, T - y_j \cup \{y_{j'}\}]$ has value $val[S,T] - n_{j'} + n_j$. Since $n_{j'} \geq n_j$ the value of the new cut is at most that of $[S,T]$. Thus for minimum cuts it is enough to consider cuts with $S \cap \{y_1, y_2, \ldots, y_q\} = \{y_1, y_2, \ldots, y_l\}$ for some l. If $x_{i'} \in S$ and $x_i \notin S$ for $i < i'$, the cut $[S - x_{i'} \cup \{x_i\}, T - x_i \cup \{x_{i'}\}]$ has value $val[S,T] + m_{i'} - m_i$. Since $m_{i'} \leq m_i$ the value of the new cut is at most that of $[S,T]$. Thus for minimum cuts it is enough to consider cuts with $S \cap \{x_1, x_2, \ldots, y_p\} = \{x_1, x_2, \ldots, y_k\}$ for some l. A cut with $S = \{s\} \cup \{x_1, x_2, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_l\}$ has value $k(q-l) + \sum_{i=k+1}^{p} m_i + \sum_{j=1}^{l} n_j$ and

$$k(q-l) + \sum_{i=k+1}^{p} m_i + \sum_{j=1}^{l} n_j \geq \sum_{i=1}^{p} m_i \Leftrightarrow k(q-l) + \sum_{j=1}^{l} n_j \geq \sum_{i=1}^{k} m_i$$

Thus all cuts (and hence the minimum cut equal to maximum flow value) are at least $\sum_{i=1}^{p} m_i$ if and only if the given condition holds.