Homework 2
Combinatorics (Math 446) Fall 2004 Lehigh University

Due Wednesday September 8

7. For the primal-dual pair of linear programming problems (primal) $\max\{\boldsymbol{cx}|A\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\}$, (dual) $\min\{\boldsymbol{yb}|\boldsymbol{y}A \ge \boldsymbol{c}, \boldsymbol{y} \ge \boldsymbol{0}\}$ prove that if the primal has a finite optimum then the dual also has a finite optimum. (Thus in the statement of the strong duality theorem we could replace 'both problems feasible' with 'if at least one of the two optima is finite'.)

8. Prove that if a digraph with n nodes and upper and lower bounds on the arcs has a circulation then it has a circulation in which the flow values are either the upper or lower bound for all but at most $n-1$ of the arcs. Prove this in a manner similar to the proof of the integrality theorem given in class. You may use the fact that any undirected graph with n vertices and at least n edges contains a cycle. (This could also be proved using results about basic solutions obtained via the simplex method but give a direct proof.)

9. Use Hoffman's circulation theorem to prove the max-flow/min-cut theorem.

10. A linear system $A'\boldsymbol{x}' \le \boldsymbol{b}', \boldsymbol{x}' \ge \boldsymbol{0}$ can be *homogenized* by adding a new positive variable z as $A'\boldsymbol{x}' - \boldsymbol{b}'z \le \boldsymbol{0}', \boldsymbol{x}' \ge \boldsymbol{0}, z > 0$. Show that the homogenized system has a solution if and only if the original does.

11. Show that any linear programming problem of the form $\max\{\boldsymbol{cx}|A\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\}$ can be reduced to a game problem. (That is, for any LP there exists a payoff matrix P whose solution can be used to solve the LP. The proof of the minimax theorem for games shows that for any game there exists an LP whose solution gives the game solution, but the LP constructed in the proof have special form so you cannot just reverse this process.)

For $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, $\boldsymbol{b}^T = (6, 7)$ and $\boldsymbol{c} = (8, 9, 0)$ give the P that results from applying the method of your proof.

Hints: Note that by symmetry if a payoff matrix is skew-symmetric ($-P^T = P$) then the value of the game is 0 as the game 'looks the same' to both players. Homogenize (as in the previous problem) the system (2) in the notes on duality. Show how to permute the resulting partitioned constraint matrix to make it skew-symmetric. The maximization problem for the game for player 2 has a solution with value 0. Assume that there is a solution in which the variable corresponding to z is positive. (It can be shown that if the primal and dual are feasible then this will be the case.) Use such a solution to give solutions to the original LP and its dual.

12. Do exercise 19.2.12 from the text.