Solutions to combinatorics (math 446) homework 1, fall 2004, Lehigh University

1. Note - there are at least two ways to take care of the 'at most one of the systems has a solution' part of the statements. While it is a bit redundant we will show both ways below. First we show it directly and we also show it by the equivalent systems. If the 'at most one system holds' is shown first then only the $\Leftarrow$'s are needed for the equivalent systems.

First we note that for each it is easy to show that at most one of the systems holds.

If both IA and IIA hold then

$$0 = \mathbf{0}\mathbf{0} \leq (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) \leq \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction. We have used $\boldsymbol{y} \geq \mathbf{0}$ in the second $\leq$.

If both IB and IIB hold then

$$0 = \mathbf{0}\mathbf{0} \leq (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) = \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction.

So for the remainder we will seek to show *at least one of the following holds.*

$(A \Rightarrow B)$: Note the following equivalences.

$$\text{(IB)} \quad \begin{array}{c} A\boldsymbol{x} = \boldsymbol{b} \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ -A\boldsymbol{x} \leq -\boldsymbol{b} \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} A \\ -A \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \quad \text{(IB')}$$

and

$$\text{(IIB)} \quad \begin{array}{c} \boldsymbol{y}A \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} (\boldsymbol{u}-\boldsymbol{v})A \geq \mathbf{0} \\ (\boldsymbol{u}-\boldsymbol{v})\boldsymbol{b} < 0 \\ \boldsymbol{u}, \boldsymbol{v} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} A \\ -A \end{bmatrix} \geq \mathbf{0} \\ \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} < 0 \\ \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \geq \mathbf{0} \end{array} \quad \text{(IIB')} \ .$$

In the second line, given $\boldsymbol{y}$ one can easily pick non-negative $\boldsymbol{u}, \boldsymbol{v}$ such that $\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{v}$ so the first $\Leftrightarrow$ in the second line does hold.

Applying A, we get that exactly one of (IB') and (IIB') has a solution. The equivalences then show that exactly one of (IB) and (IIB) has a solution.

$(B \Rightarrow A)$: Note following equivalences.

$$\text{(IA)} \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} + I\boldsymbol{s} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{s} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{b} \\ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} \geq \mathbf{0} \end{array} \quad \text{(IA')}$$

and

$$\text{(IIA)} \quad \begin{array}{c} \boldsymbol{y}A \geq \mathbf{0} \\ \boldsymbol{y} \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y}A \geq \mathbf{0} \\ \boldsymbol{y}I \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y}\left[\begin{array}{cc} A & I \end{array}\right] \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \text{(IIA')}\ .$$

Applying B, we get that exactly one of (IA') and (IIA') has a solution. The equivalences then show that exactly one of (IA) and (IIA) has a solution.

2. (i) We start with

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \text{ for } i = 1, 2 \ldots, m. \tag{1}$$

Let $U = \{i | a_{in} > 0\}$, $L = \{i | a_{in} < 0\}$ and $N = \{i | a_{in} = 0\}$. Then

$$\begin{array}{l} \dfrac{1}{a_{rn}}\left(b_r - \displaystyle\sum_{j=1}^{n-1} a_{rj}x_j\right) \leq x_n \text{ for } r \in L \\ x_n \leq \dfrac{1}{a_{sn}}\left(b_s - \displaystyle\sum_{j=1}^{n-1} a_{sj}x_j\right) \text{ for } s \in U \\ \displaystyle\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \text{ for } t \in N \end{array} \tag{2}$$

is just a rearrangement of (1). Note that the direction of the inequality changes when $r \in L$ since $a_{rn} < 0$ and we multiply by this. We pair each upper bound with each lower bound and carry along the inequalities not involving x_n to get

$$\begin{array}{l} \dfrac{1}{a_{rn}}\left(b_r - \displaystyle\sum_{j=1}^{n-1} a_{rj}x_j\right) \leq \dfrac{1}{a_{sn}}\left(b_s - \displaystyle\sum_{j=1}^{n-1} a_{sj}x_j\right) \text{ for } r \in L, s \in U \\ \displaystyle\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \text{ for } t \in N \end{array} \tag{3}$$

Due to the construction (1) has a solution if and only if (3) does. (This will also follow from parts (iv) and (v) below.)

(ii) With one variable we have three types of inequalities as above using $n = 1$ for L, U, N: $a_{s1}x_1 \leq b_s$ for $s \in U$; $a_{r1}x_1 \leq b_r$ for $r \in L$ and inequalities with no variable ($t \in N$) of the form $0 \leq b_t$. If we have $0 \leq b_t$ for some $b_t < 0$ then the system is inconsistent, there is clearly no solution to the system. Set all multiplies $u_{i1} = 0$ except $u_{t1} = 1$ to get a certificate of inconsistency. We can drop inequalities $0 \leq b_t$ for $b_t \geq 0$ so we can assume now that all a_{i1} are nonzero. Rewriting the system we have $x_1 \leq b_s/a_{s1}$ for $s \in U$ and $b_r/a_{r1} \leq x_1$ for $r \in L$. There is a solution if and only if $\max_{r \in L} b_r/a_{r1} \leq \min_{s \in U} b_s/a_{s1}$. If this holds then any $x_1 \in [\max_{r \in L} b_r/a_{r1}, \min_{s \in U} b_s/a_{s1}]$ satisfies all inequalities. If not then for some r^*, s^* we have $b_{s^*}/a_{s^*1} < b_{r^*}/a_{r^*1}$. For a certificate of inconsistency take all $u_{i1} = 0$ except $u_{r^*1} = a_{s^*1} > 0$ and $u_{s^*1} = -a_{r^*1} > 0$. Multiplying we have $a_{s^*1}\left(a_{r^*1}x_1 \leq b_{r^*}\right)$ and $-a_{r^*1}\left(a_{s^*1}x_1 \leq b_{s^*}\right)$. Combining these we get $0 \leq \left(b_{r^*}a_{s^*1} - b_{s^*}a_{r^*1}\right) < 0$. The last < 0 follows from $b_{s^*}/a_{s^*1} < b_{r^*}/a_{r^*1}$ (again noting that the direction of the inequality changes as $s_{s^*1} < 0$).

(iii) If $L \cup N$ is empty the original system is $x_n \leq \dfrac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)$ for $s \in U$. For a solution take $x_1 = x_2 = \cdots x_{n-1} = 0$ and $x_n = \min_{s\in U}(b_s/a_{sn})$. It is straightforward to check that this is a solution.

If $U \cup N$ is empty the original system is $\dfrac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq x_n$ for $r \in L$. For a solution take $x_1 = x_2 = \cdots x_{n-1} = 0$ and $x_n = \max_{r\in L}(b_r/a_{rn})$. It is straightforward to check that this is a solution.

(iv) If (3) is inconsistent with multipliers u_{rs} for $r \in L, s \in U$ and u_t for $t \in N$ construct a certificate $\boldsymbol{y}$ of inconsistency for (1): For $t \in N$ let $y_t = u_t$. For $r \in L$ let $y_r = -\dfrac{1}{a_{rn}} \sum_{s\in U} u_{rs}$ and for $s \in U$ let $y_s = \dfrac{1}{a_{sn}} \sum_{r \in L} u_{rs}$. Since we started with a certificate for (3) we have that combining the inequalities

$$\begin{array}{ll} u_{rs}\left(\dfrac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq \dfrac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)\right) & \text{for } r \in L, s \in U \\ u_t\left(\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t\right) & \text{for } t \in N \end{array}$$

yields $0 < b$ for some $b < 0$. For (2) which is a rearrangement of (1), combining

$$\begin{array}{l} \left(-\dfrac{1}{a_{rn}}\sum_{s\in U} u_{rs}\right)\left(\dfrac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq x_n\right) \text{ for } r \in L \\ \left(\dfrac{1}{a_{sn}}\sum_{r\in L} u_{rs}\right)\left(x_n \leq \dfrac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)\right) \text{ for } s \in U \\ (u_t)\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \text{ for } t \in N \end{array}$$

(v) Given a solution $(x_1^*, x_2^*, \ldots, x_{n-1}^*)$ to (3) take x_n^* to be any value in the interval

$$\left[\max r \in L \frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right), \min s \in U \frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)\right]$$

3. We will eliminate the variable x_1. Rewriting we get the following 'equivalent' systems (where equivalent means either both have solutions or both do not):

For (a)

$$\begin{array}{rcl} 3x_1 + 6x_2 & \leq & 12 \\ -x_1 + x_2 & \leq & 2 \\ -2x_1 - 6x_2 & \leq & -10 \end{array} \Leftrightarrow \begin{array}{rcl} x_1 & \leq & 4 - 2x_2 \\ -2 + x_2 & \leq & x_1 \\ 5 - 3x_2 & \leq & x_1 \end{array} \Leftrightarrow \begin{array}{rcl} -2 + x_2 & \leq & 4 - 2x_2 \\ 5 - 3x_2 & \leq & 4 - 2x_2 \end{array} \Leftrightarrow \begin{array}{rcl} 3x_2 & \leq & 6 \\ -x_2 & \leq & -1 \end{array}$$

The last system is $1 \le x_2 \le 2$. We can pick x_2 in this interval and use it to get a solution for x_1. Take $x_2 = 3/2$. (The arithmetic for taking $x_2 = 1$ or $x_2 = 2$ is simpler but we happen get only one possibility for x_1 so we take $x_2 = 3/2$ for better illustration.) Putting $x_2 = 3/2$ into the original system gives $x_1 \le 4 - 2(3/2) = 1$, $-1/2 = -2 + 3/2 \le x_1$, $1/2 = 5 - 3(3/2) \le x_1$. So any x_1 in the interval $[1/2, 1]$ along with $x_2 = 3/2$ is a solution. In particular a solution is $x_1 = 1/2, x_2 = 3/2$.

For (b)

$$\begin{array}{lcl} 3x_1 + 6x_2 & \le & 12 \\ -x_1 + x_2 & \le & 2 \\ -2x_1 - 6x_2 & \le & -14 \end{array} \Leftrightarrow \begin{array}{lcl} x_1 & \le & 4 - 2x_2 \\ -2 + x_2 & \le & x_1 \\ 7 - 3x_2 & \le & x_1 \end{array} \Leftrightarrow \begin{array}{lcl} -2 + x_2 \le & 4 - 2x_2 & \\ 7 - 3x_2 & \le & 4 - 2x_2 \end{array} \Leftrightarrow \begin{array}{lcl} 3x_2 \le & 6 & \\ -x_2 & \le & -3 \end{array}$$

The last system is inconsistent as seen by the multipliers $(1,3)$. (Multiply the first inequality by 1 and the second by 3 and combine to get $0 \le -3$.) This correspond to multipliers $(1,3)$ in the third system, multipliers $(4,1,3)$ in the second system and $(4/3, 1, 3/2)$ in the original, yielding the same inconsistency $0 \le -3$.

4. Use induction on the size of the submatrix. A 1 by 1 submatrix is an entry of A which is in $\{0, -1, +1\}$ and the determinant is trivially the same. Given a k by k submatrix B, consider three cases: (a) Some column has all zeroes; (b) some column has exactly one $+1$ or exactly one -1; (c) not (a) or (b).

Then, if:
(a) $det(B) = 0$ as it has a column of all zeroes.
(b) Say column s has exactly one non-zero and the non-zero is in row r. Then expanding the determinant we get $Det(B) = \sum_{i=1}^{k} a_{ij} C(ij) = a_{rs} C(rs)$. This follows since $a_{is} = 0$ except when $i = r$. Finally, $a_{rs} = \pm 1$ and $C(rs)$ is ± 1 times a smaller submatrix of A, which by induction is in $\{0, -1, +1\}$. Thus $det(B)$ is a product of three terms all in $\{0, -1, +1\}$ and hence $det(B) \in \{0, -1, +1\}$.
(c) In this case all columns of B have exactly one $+1$ and exactly one -1. Summing all rows gives a row of all zeroes and hence $det(B) = 0$.

5. (See also the hand drawn diagrams.) The digraph D' has the following arcs with weight $-\epsilon$: $\{l_v r_u, l_w r_u, l_y r_u, l_w r_v, l_w r_x, l_y r_x, l_w r_y\}$ and the following two sets each with arc weights 0: $\{r_x l_u, r_u l_x, r_x l_v, r_v l_x, r_y l_v, r_v$ and $\{r_u l_u, r_v l_v, r_w l_w, r_x l_x, r_y l_y\}$. A potential (found by inspection) is $p(r_u) = p(l_u) = p(l_x) = -3\epsilon$, $p(l_v) = p(r_x) = -2\epsilon$, $p(r_y) = p(l_y) = p(r_v) = -\epsilon$, $p(r_w) = p(l_w) = 0$. After scaling by ϵ this gives the interval representation $I_u = [-3, -3]$, $I_v = [-2, -1]$, $I_w = [0, 0]$, $I_x = [-3, -2]$, $I_y = [-1, -1]$.

6. (See also the attached hand drawn diagrams.) We get the a digraph with vertices $\{0, 1, \ldots, 7\}$ and arcs ij for all $1 \le i < j \le 7$ with lower bound 0 and upper bound 1 and the following arcs and bounds: 01 with $l(01) = u(01) = 6 - 1 + 1 = 6$, 02 with $l(02) = u(02) = 5 - 2 + 1 = 4$, 03 with $l(03) = u(03) = 5 - 3 + 1 = 3$, 04 with $l(04) = u(04) = 3 - 4 + 1 = 0$, 05 with $l(05) = u(05) = 1 - 5 + 1 = -3$, 06 with $l(06) = u(06) = 1 - 6 + 1 = -4$, 07 with $l(07) = u(07) = 0 - 7 + 1 = -6$. The set $\{4, 5, 6, 7\}$ has sum of lower bounds on arcs leaving equal to 0 (in fact it has no arcs leaving) and sum of entering upper bounds equal to $12 + (0 + (-3) + (-4) + (-6)) = -1$. (The 12 comes from upper bounds of 1 on the 12 arcs from $\{1, 2, 3\}$ to $\{4, 5, 6, 7\}$ and the remaining terms are from upper bounds on the arcs $04, 05, 06, 07$. Thus at most -1 unit of flow can enter $\{4, 5, 6, 7\}$ and

at least 0 units leave. This violates the necessary condition for flows. Then, in the score sequence s_4, s_5, s_6, s_7 violate the necessary condition with $5 = 3 + 1 + 1 + 0 = s_4 + s_5 + s_6 + s_7 < 6 = \binom{4}{2}$. (Note $\{5, 6, 7\}$ also violate the flow condition and s_5, s_6, s_7 violate the score condition.)