Homework 1
Combinatorics (Math 446) Fall 2004 Lehigh University (updated version)

Due Wednesday September 1

1. Prove the equivalence of A and B:

A: Exactly one of the following holds:

(I) $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$
(II) $\boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0$ has a solution $\boldsymbol{y}$

B: Exactly one of the following holds:

(I) $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$
(II) $\boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0$ has a solution $\boldsymbol{y}$

2. Consider the system $A\boldsymbol{x} \leq \boldsymbol{b}$ which can also be written as $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for $i = 1, 2 \ldots, m$. Let $U = \{i | a_{in} > 0\}$, $L = \{i | a_{in} < 0\}$ and $N = \{i | a_{in} = 0\}$. Outline a proof of Farkas' Lemma as follows:
(i) Describe, using the notation above, a system with variables $x_1, x_2, \ldots, x_{n-1}$ that has a solution if and only if the original system does. (This should be a generic version of the Fourier-Motzkin example.)
(ii) Show that Farkas' lemma holds for systems with 1 variable. (This is 'obvious', the exercise is to come up with appropriate notation to make it a real proof. One could also use the no variable case as a basis for induction but notation is more complicated for this.)
(iii) If the system in (i) is empty show that the original system has a solution. (It is empty if $L \cup N$ or $U \cup N$ is empty.)
(iv) Assume the system in (i) is inconsistent and multipliers u_{rs} for $r \in L, s \in U$, v_t for $t \in N$ provide a certificate of inconsistency. Describe a certificate of inconsistency for the original system.
(v) Assume the system in (i) has a solution $x_1^*, x_2^*, \ldots, x_{n-1}^*$. Describe the set of solutions to the original problem that agrees with the x_j^* for $j = 1, 2, \ldots n-1$.

3. Apply Fourier-Motzkin elimination to the following systems. Using this method, either determine one solution or give a certificate showing the original system is inconsistent. Use Fourier-Motzkin elimination to get your answers, showing how it could work on much larger systems.

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\textbf{(a)}\quad \begin{array}{rcrcr} 3x_1 & + & 6x_2 & \le & 12 \\ -x_1 & + & x_2 & \le & 2 \\ -2x_1 & - & 6x_2 & \le & -10 \end{array}
\qquad
\textbf{(b)}\quad \begin{array}{rcrcr} 3x_1 & + & 6x_2 & \le & 12 \\ -x_1 & + & x_2 & \le & 2 \\ -2x_1 & - & 6x_2 & \le & -14 \end{array}
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4. Let A be a matrix with entries $0, +1, -1$ with at most one $+1$ and at most one -1 in each column. Prove that A is totally unimodular (every square submatrix has determinant 0, +1, or -1). Use induction. Recall the following facts from linear algebra: (i) If a linear combination of the rows of a square matrix sums to the zero vector then the matrix is singular and thus has determinant 0. (ii) If a row or column has all zero entries then the determinant is 0. (This is really just a special case of (i).) (iii) Recall also the formula for expanding a determinant using cofactors: $Det(A) = \sum_{i=1}^{n} a_{ij} C(ij)$ where the cofactor $C(ij)$ is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting row i and column j.

5. Let D be the transitive acyclic digraph with vertices $\{u, v, w, x, y\}$ and arcs $uv, uw, vw, uy, xy, yw, xw$. Construct the new digraph D' for determining if D is an interval digraph. Use a potential in D' to construct an interval representation for D.

6. For the possible score sequence $\boldsymbol{s} = (6, 5, 5, 3, 1, 1, 0)$ construct the digraph D for determining if $\boldsymbol{s}$ is a score sequence. Show that D does not have a circulation and use this to get a set violating the necessary condition for a score sequence.