Math 446 Combinatorics Exam 1 Lehigh University 9 - 27 - 04

This is closed book, closed notes etc.
You have 75 minutes to take this exam.
There are 10 problems each counting 10 points.
Explain your answers carefully and completely.

**1:** Prove that if a given network with integral upper and lower bounds for the flow on each arc has a feasible flow then it has one with integral flow values. This is the integrality theorem. Give a direct proof, involving modification of a feasible flow.

**2:** The statement of the max-flow min-cut theorem begins with 'In every network having a feasible flow...' In the case that there are non-trivial lower bounds on the flow on some arcs it is possible that there is no feasible flow. Make use of the circulation theorem to find and prove a necessary and sufficient condition for the existence of a feasible $s-t$ flow. Assume that lower bounds may be negative. Hint - your conditions will not be exactly the same as the circulation conditions but may be very similar to them. Be very careful how you state the conditions.

**3:** Consider the primal-dual pair of linear programming problems
(primal) $\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, (dual) $\min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}\}$ prove that if the primal is feasible and has a finite optimum then the dual is feasible and has a finite optimum.

**4:** Prove the equivalence of the strong duality theorems for the following primal-dual pairs:
$\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$ and
$\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}\}$.

**5:** (i) Prove the following weak duality theorem:
$\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} \leq \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$.
(ii) Describe a system of linear inequalities whose solution would provide optimal solutions to both the primal and dual of part (i).

**6:** Recall the Konig-Egevary Theorem that the maximum size of a matching is equal to the minimum size of a vertex cover in a bipartite graph. Prove this using a linear programming approach.

**7:** Recall Dilworth's Theorem: In a partially ordered set (a transitive digraph), the maximum size of an antichain is equal to the minimum number of chains needed to cover the elements. Prove this in any manner you choose.

**8:** Consider $\begin{array}{rcrcrcr} 3x_1 & - & 9x_2 & + & 6x_3 & \leq & 12 \\ -4x_1 & + & 4x_2 & + & 8x_3 & \leq & -12 \\ 2x_1 & + & 6x_2 & - & 20x_3 & \leq & 0 \end{array} \Leftrightarrow \begin{array}{rcrcr} -2x_2 & + & 4x_3 & \leq & 1 \\ 4x_2 & - & 8x_3 & \leq & -3 \end{array}.$

By $\Leftrightarrow$ we mean that the system on the left has a solution if and only if the system on the right does. The system on the right is obtained using Fourier-Motzkin elimination, eliminating the variable $x_1$. Show/explain how this is done. That is show several steps of the process to eliminate $x_1$. The system on the right is inconsistent with certificate $(2,1)$; multiply the first row by 2 and the second by 1 and combine. Use this certificate to obtain a certificate of inconsistency for the system on the left.

**9:** Several companies send representatives to a conference; the $i^{th}$ company sends $m_i$ representatives. The organizers of the conference conduct simultaneous networking groups; the $j^{th}$ group can accommodate up to $n_j$ participants. The organizers want to schedule all of the participants into groups, but participants from the same company must be in different groups. The groups need not all be filled. Design a network test to see whether the constraints can be satisfied and prove that there exists an assignment of participants to groups that satisfies all of the constraints if and only if for all $0 \leq k \leq p$ and $0 \leq l \leq q$ we have $k(q-l) + \sum_{j=1}^{l} n_j \geq \sum_{i=1}^{k} m_i$. Here we assume $p$ companies and $q$ groups with $m_1 \geq m_2 \geq \cdots \geq m_p$ and $n_1 \leq n_2 \leq \cdots \leq n_q$.

**10:** Let $\sigma$ and $\pi$ be permutations of $k-$element subsets of $n$. Say that $\sigma$ and $\pi$ intersect if $\sigma(i) = \pi(i)$ for some $i$. Prove that the maximum size of a pairwise intersecting family of such $k-$permutations of $[n] = \{1, 2, \ldots, n\}$ is $(n-1)!/(n-k)!$.