Discrete-Time Signal Analysis in the Frequency-Domain

Eugenio Schuster
Lehigh University
Mechanical Engineering and Mechanics
schuster@lehigh.edu

Tutorial

1 Frequency Response of Discrete-Time LTI Systems

For a linear time-invariant (LTI) system with impulse response $h[n]$, the output sequence $y[n]$ is related to the input sequence $u[n]$ through the convolution sum,

$$y[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k], \tag{1}$$

where n is an integer number. Consider as an input sequence a complex exponential of radian frequency ω, i.e. $u[n] = e^{j\omega n}$ for $-\infty < n < \infty$. The output of the system is given by

$$y[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}. \tag{2}$$

Defining,

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}, \tag{3}$$

we can write the output sequence as

$$y[n] = H(e^{j\omega})e^{j\omega n}. \tag{4}$$

As result we have that the complex exponential $e^{j\omega n}$ is an eigenfunction of the LTI system with associated eigenvalue equal to $H(e^{j\omega})$. The eigenvalue $H(e^{j\omega})$ is called the *Frequency Response* of the system and describes the changes in amplitude and phase of the complex exponential input. An important distinction exists between continuous-time and discrete-time LTI systems. While in the continuous-time domain we need specify the frequency response $H(\Omega)$ over the interval $-\infty < \Omega < \infty$, in the discrete-time domain we only need specify the frequency response $H(e^{j\omega})$ over an interval of length 2π, e.g., $-\pi < \omega \leq \pi$. This property is based on the periodicity of the complex exponential. Using the fact that $e^{\pm j2\pi r} = 1$ for any integer r, we can show that

$$e^{-j(\omega+2\pi r)n} = e^{-j\omega n}e^{-j2\pi rn} = e^{-j\omega n}. \tag{5}$$

As we will show later, a broad class of input signals can be represented by a linear combination of complex exponentials. In this case, the knowledge of the frequency response allows us to find the output of the LTI system.

2 Discrete-Time Fourier Transform

Many sequences can be represented by a Fourier integral of the form

$$u[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\omega})e^{j\omega n}d\omega, \tag{6}$$

where

$$U(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n]e^{-j\omega n}. \tag{7}$$

The *Inverse Fourier Transform* (6) represents $u[n]$ as a superposition of infinitesimal complex exponentials over the interval $-\pi < \omega \leq \pi$. The *Discrete-Time Fourier Transform* (7), or simply

Fourier Transform in this tutorial, determines how much of each frequency component over the interval $-\pi < \omega \leq \pi$ is required to synthesize $u[n]$ using eq. (6). The Fourier Transform is usually referred as *Spectrum*. Comparing eqs. (3) and (7), it is possible to note that the frequency response of a LTI system is the Fourier Transform of the impulse response $h[n]$. As we stated above, the frequency response is periodic. Likewise, the Fourier Transform is periodic with period 2π.

3 Z-Tranform

Given a sequence $u[n]$, we define the *Z-Transform* as

$$U(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n}. \tag{8}$$

Comparing eqs. (7) and (8) we note that we can obtain the Fourier Transform evaluating the Z-Transform at the unit circle ($z = e^{j\omega}$). Based on this property, the frequency response $H(e^{j\omega})$ of a discrete-time LTI system $h[n]$ can be obtained evaluating the Z-Transform $H(z)$ at $z = e^{j\omega}$.

4 Relationship between sequences and sampled signals

A sequence $u[n]$ is generally a representation of a sampled signal. Given a continuous signal $u(t)$, its sampled version $u_s(t)$ can be written as $u_s(t) = u(t)s(t)$ with

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \tag{9}$$

where δ is the Dirac delta function and T_s is the sampling period. In this case we write

$$u_s(t) = u(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} u(nT_s)\delta(t - nT_s), \tag{10}$$

and

$$u[n] = u(nT_s). \tag{11}$$

Based on the definition of the *Continuous-Time Fourier Transform*,

$$u[t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\Omega)e^{j\Omega t} d\Omega \tag{12}$$

$$U(\Omega) = \int_{-\infty}^{\infty} u(t)e^{-j\Omega t} dt, \tag{13}$$

we can obtain the continuous-time spectrum for the sampled signal

$$U_s(\Omega) = \sum_{n=-\infty}^{\infty} u(nT_s) \int_{-\infty}^{\infty} \delta(t - nT_s)e^{-j\Omega t} dt = \sum_{n=-\infty}^{\infty} u(nT_s)e^{-j\Omega nT_s}. \tag{14}$$

Using the Discrete-Time Fourier Transform (7) we can compute

$$U(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n]e^{-j\omega n}. \tag{15}$$

Comparing eqs. (14) and (15), and taking into account eq. (11) we conclude that

$$U_s(\Omega) = U(e^{j\omega})\Big|_{\omega=\Omega T_s} = U(e^{j\Omega T_s}). \tag{16}$$

The Discrete-Time Fourier Transform $U(e^{j\omega})$ is simply a frequency-scaled version of the Continuous-Time Fourier Transform $U_s(\Omega)$ where the scale factor is given by

$$\omega = \Omega T_s = \frac{\Omega}{f_s} = 2\pi \frac{f}{f_s}. \tag{17}$$

Nyquist theorem relates the sampling frequency $f_s = 1/Ts$ with the maximum frequency f_{max} of the signal before sampling. In order to avoid aliasing distortion, it is required that

$$f_s > 2 f_{max}. \tag{18}$$

Therefore, every time we sample with frequency f_s we are assuming that the maximum frequency of the signal to be sampled is less than $f_s/2$. In other words, we are assuming that

$$U(\Omega) = \begin{cases} \neq 0 & -2\pi\frac{f_s}{2} < \Omega \leq 2\pi\frac{f_s}{2} \\ = 0 & \text{otherwise.} \end{cases} \tag{19}$$

Based on the scaling (17), we will have

$$U(e^{j\omega}) = \begin{cases} \neq 0 & -\pi < \omega \leq \pi \\ = 0 & \text{otherwise.} \end{cases} \tag{20}$$

implying that the interval $-\pi < \omega \leq \pi$ in the discrete-time domain corresponds to the interval $-\pi f_s < \Omega \leq \pi f_s$ $(-f_s/2 < f \leq f_s/2)$ in the continuous-time domain.

5 Discrete Fourier Series

We come back now to the idea of representing signals by a linear combination of complex exponential and we consider at this time the periodic sequence $\tilde{u}[n]$ with period N, i.e. $\tilde{u}[n] = \tilde{u}[n + rN]$ for any integer r. In this case, as in the continuous case, we can represent $\tilde{u}[n]$ by its Fourier Series,

$$\tilde{u}[n] = \frac{1}{N} \sum_k \tilde{U}[k] e^{j\frac{2\pi}{N}kn}. \tag{21}$$

By the Fourier Series, the periodic sequence is represented as a sum of complex exponentials with frequencies that are integer multiples of the fundamental frequenciy $2\pi/N$. We say that these are harmonically related complex exponentials. The Fourier Series representing continuous-time periodic signals require an infinite number of harmonically related complex exponentials, whereas the Fourier Series for any discrete-time periodic signal requires only N harmonically related complex exponentials. This is explained by the periodicity of the complex exponential,

$$e^{j\frac{2\pi}{N}(k+rN)n} = e^{j\frac{2\pi}{N}kn} e^{j\frac{2\pi}{N}rNn} = e^{j\frac{2\pi}{N}kn} \tag{22}$$

for any integer r. Thus, the *Discrete Fourier Series* of the periodic sequence $\tilde{u}[n]$ with period N can be written as

$$\tilde{u}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{U}[k] e^{j\frac{2\pi}{N}kn}, \tag{23}$$

where the Fourier Series coefficients have the form

$$\tilde{U}[k] = \sum_{n=0}^{N-1} \tilde{u}[n] e^{-j\frac{2\pi}{N}kn}. \tag{24}$$

We can note that the sequence $\tilde{U}[k]$ is periodic with period N.

6 Representing periodic sequences by Fourier Transform

We are wondering now how we can represent periodic sequences by the *Fourier Transform.* To give ourselves an answer, we must study the convergence of the infinite sum (7). A sufficient condition for convergence can be found as follows:

$$\left|U(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} u[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |u[n]| \left|e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |u[n]| < \infty \tag{25}$$

Those sequences that satisfy

$$\sum_{n=-\infty}^{\infty} |u[n]| < \infty \tag{26}$$

are called absolutely summable. When the sequence $u[n]$ is absolutely summable, the Fourier Transform $U(e^{j\omega})$ not only exists but converges uniformly to a continuos function of ω. For those sequences that are not absolutely summable but square summable, i. e.,

$$\sum_{n=-\infty}^{\infty} |u[n]|^2 < \infty \tag{27}$$

the Fourier Transform $U(e^{j\omega})$ also exists but the convergence condition is relaxed to mean-square convergence. This means that given

$$U(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n]e^{-j\omega n} \tag{28}$$

and

$$U_M(e^{j\omega}) = \sum_{n=-M}^{M} u[n]e^{-j\omega n} \tag{29}$$

it follows that

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \left|U(e^{j\omega}) - U_M(e^{j\omega})\right|^2 d\omega = 0, \tag{30}$$

which means that total "energy" of the error approaches zero as $M \to \infty$. In summary,

Uniform Convergence of the Fourier Transform	$\Leftarrow$	Sequence is absolutely summable
Mean-Square Convergence of the Fourier Transform	$\Leftarrow$	Sequence is square summable.

The periodic sequence $\tilde{u}[n]$ satisfies neither (26) nor (27). However, sequences that can be expressed as a sum of complex exponentials, as it is the case for all periodic sequences, can be considered to have a Fourier Transform as a train of impulses. It is simple to demonstrate that the expression

$$U(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi r), \tag{31}$$

where we assume that $-\pi < \omega_o \leq \pi$, corresponds to the Fourier Transform of the complex exponential sequence $e^{j\omega_o n}$. To show that, we replace the expression in (6) to obtain

$$u[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\omega})e^{j\omega n}d\omega = \int_{-\pi}^{\pi} \delta(\omega - \omega_o)e^{j\omega n}d\omega = e^{j\omega_o n}. \tag{32}$$

Then, if a sequence $u[n]$ can be represented as a sum of complex exponentials, i. e.,

$$u[n] = \sum_k a_k e^{j\omega_k n} \tag{33}$$

for $\infty < n < \infty$, it has a Fourier Transform given by

$$U(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k a_k 2\pi\delta(\omega - \omega_k + 2\pi r). \tag{34}$$

This means that if $\tilde{u}[n]$ is periodic with period N and Discrete Fourier Series coefficients $\tilde{U}[k]$, we can write

$$\tilde{u}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{U}[k]e^{j\frac{2\pi}{N}kn}, \tag{35}$$

and the Fourier Transform $\tilde{U}(e^{j\omega})$ is defined to be the impulse train

$$\tilde{U}(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_{k=0}^{N-1} 2\pi\frac{\tilde{U}[k]}{N}\delta(\omega - \frac{2\pi k}{N} + 2\pi r) = \sum_{k=-\infty}^{\infty} 2\pi\frac{\tilde{U}[k]}{N}\delta(\omega - \frac{2\pi k}{N}). \tag{36}$$

As an example, consider now a periodic impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1 & n = rN \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

where r is an integer and N is the period. We can compute first the Discrete Fourier Series coefficients

$$\tilde{P}[k] = \sum_{n=0}^{N-1} \tilde{p}[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \delta[n]e^{-j\frac{2\pi}{N}kn} = 1. \tag{38}$$

Therefore, according to (36) the Fourier Transform is given by

$$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta(\omega - \frac{2\pi k}{N}). \tag{39}$$

The Fourier Transform of the periodic impulse train becomes important when we want to relate finite-length and periodic sequences.

7 Finite-length sequences as periodic sequences

Consider now a finite length sequence $u[n]$ ($u[n] = 0$ everywhere except over the interval $0 \leq n \leq N-1$). We can construct an associated periodic sequence $\tilde{u}[n]$ as the convolution of the finite-length sequence with the impulse train (37) of period N:

$$\tilde{u}[n] = u[n] * \tilde{p}[n] = u[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} u[n - rN]. \tag{40}$$

The periodic sequence $\tilde{u}[n]$ is a set of periodically repeated copies of the finite-length sequence $u[n]$. Assuming that the Fourier Transform of $u[n]$ is $U(e^{j\omega})$, and recalling that the Fourier Transform of a convolution is the product of the Fourier Transforms, we can obtain the Fourier Tranform for $\tilde{u}[n]$ as

$$\tilde{U}(e^{j\omega}) = U(e^{j\omega})\tilde{P}(e^{j\omega}) = U(e^{j\omega}) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta(\omega - \frac{2\pi k}{N}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} U(e^{j\frac{2\pi k}{N}})\delta(\omega - \frac{2\pi k}{N}), \tag{41}$$

where we have used (39). This result must be coincident with our definition (36) and therefore it must be

$$\tilde{U}[k] = U(e^{j\frac{2\pi k}{N}}) = U(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} . \tag{42}$$

This very important result implies that the periodic sequence $\tilde{U}[k]$ with period N of Discrete Fourier Series coefficients are equally spaced samples of the Fourier Transform of the finite-length sequence $u[n]$ obtained by extracting one period of $\tilde{u}[n]$. This corresponds to sample the Fourier Transform at N equally spaced frequencies over the interval $-\pi < \omega \leq \pi$ with spacing $2\pi/N$.

8 Discrete Fourier Transform

As we defined

$$u[n] = \begin{cases} \tilde{u}[n] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

$$\tilde{u}[n] = u[(n \text{ modulo } N)] \tag{44}$$

we define now for consistency (and to mantain duality between time and frequency),

$$U[k] = \begin{cases} \tilde{U}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

$$\tilde{U}[k] = U[(k \text{ modulo } N)]. \tag{46}$$

We have used the fact that the Discrete Fourier Series sequence $\tilde{U}[k]$ is itself a sequence with period N. The sequence $U[k]$ is named *Discrete Fourier Transform* and is written as

$$U[k] = \sum_{n=0}^{N-1} u[n] e^{-j\frac{2\pi}{N}kn}, \tag{47}$$

$$u[n] = \frac{1}{N}\sum_{k=0}^{N-1} U[k] e^{j\frac{2\pi}{N}kn}. \tag{48}$$

The *Discrete Fourier Transform* (48) gives us the *Discrete-Time Fourier Transform* (or simply the *Fourier Transform*) (7) at N equally spaced frequencies over the interval $0 \le \omega \le 2\pi$ (or $-\pi < \omega \le \pi$):

$$U[k] = U(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}, \quad 0 \le k \le N-1. \tag{49}$$

In addition to its theoretical importance as a Fourier representation of sequences, the Discrete Fourier Transform (DFT) plays a central role in digital signal processing because there exist efficient algorithms for its computation. These algorithms are usually referred as Fast Fourier Transform (FFT). The FFT is simply an efficient implementation of the DFT. In applications based on Fourier analysis of signals, it is the Discrete-Time Fourier Transform (FT) that is desired, while it is the Discrete Fourier Transform (DFT) that is actually computed. For finite-length signals, the DFT provides frequency-domain samples of the FT and the implications of this sample must be clearly understood and accounted for. Given a sequence $u[n]$ of N points sampled at frequency f_s, we can compute the DFT via the FFT as

$$U[k] = \sum_{n=0}^{N-1} u[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \tag{50}$$

$$u[n] = \frac{1}{N}\sum_{k=0}^{N-1} U[k] e^{j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1. \tag{51}$$

Considering the relationship (49) between DFT and FT, we can write the spectrum as

$$U(\omega) = U(e^{j\omega}) = \sum_{n=0}^{N-1} u[n] e^{-j\omega n}, \quad \omega = \frac{2\pi}{N}k \quad (0 \le k \le N-1; 0 \le \omega < 2\pi). \tag{52}$$

Considering the scaling (17) between sequences and sampled signals, we can write the spectrum as

$$U(\Omega) = = \sum_{n=0}^{N-1} u[n] e^{-j\Omega T_s n}, \quad \Omega = \frac{2\pi k}{N T_s} \quad \left(0 \le k \le N-1; 0 \le \Omega < \frac{2\pi}{T_s} = 2\Omega_{max}\right) \tag{53}$$

$$U(f) = = \sum_{n=0}^{N-1} u[n] e^{-j\frac{2\pi f}{f_s} n}, \quad f = \frac{k f_s}{N} \quad (0 \le k \le N-1; 0 \le f < f_s = 2 f_{max}) \tag{54}$$

In addition to the periodicity of the DFT ($U(\omega + 2\pi) = U(\omega)$), we have that $U(-\omega) = \overline{U(\omega)}$ for real $u[n]$. Therefore, the function $U(\omega)$ is uniquely defined by its values over the interval $[0, \pi]$. We associate high frequencies with frequencies close to π and low frequencies with frequencies close to 0. As a consequence of these properties, it is exactly the same to define $U(\omega)$ over the interval $0 \le \omega < 2\pi$ or the interval $-\pi < \omega \le \pi$. The DFT will give the values of the FT over any of these intervals with a frequency spacing equal to $2\pi/N$.

9 Discrete-Time Random Signals

Until now, we have assumed that the signals are deterministic. Sometimes, the mechanism of signal generation is so complex that it is very difficult, if not impossible, to represent the signal as deterministic. In these cases, modeling the signal as an outcome of a random variable is extremely useful. Each individual sample $u[n]$ of a particular signal is assumed to be an outcome of a random variable $\mathbf{u}_n$. The entire signal is represented by a collection of such random variables, one for each sample time, $-\infty < n < \infty$. The collection of these random variables is called a random process. We assume that the sequence $u[n]$ for $-\infty < n < \infty$ is generated by a random process with specific probability distribution that underlies the signal.

An individual random variable $\mathbf{u}_n$ is described by the probability distribution function

$$F_{\mathbf{u}_n}(u_n, n) = \text{Probability}[\mathbf{u}_n \leq u_n], \tag{55}$$

where $\mathbf{u}_n$ denotes the random variable and u_n is a particular value of $\mathbf{u}_n$. If $\mathbf{u}_n$ takes on a continuous range of values, it can be specified by the probability density function

$$f_{\mathbf{u}_n}(u_n, n) = \frac{\partial F_{\mathbf{u}_n}(u_n, n)}{\partial u_n} \tag{56}$$

$$F_{\mathbf{u}_n}(u_n, n) = \int_{-\infty}^{u_n} f_{\mathbf{u}_n}(u, n) du. \tag{57}$$

When we have two stochastic processes $\mathbf{u}_n$ and $\mathbf{v}_n$, the interdependence is described by the joint probability distribution function

$$F_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) = \text{Probability}[\mathbf{u}_n \leq u_n \text{ and } \mathbf{v}_m \leq v_m], \tag{58}$$

and by the joint probability density

$$f_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) = \frac{\partial^2 F_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m)}{\partial u_n \partial v_m}. \tag{59}$$

When $F_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) = F_{\mathbf{u}_n}(u_n, n) F_{\mathbf{v}_m}(v_m, m)$ we say that the processes are independent.

It is often useful to characterize a random variable in terms of its mean, variance and autocorrelation. The mean of a random process $\mathbf{u}_n$ is defined as

$$m_u[n] = m_{\mathbf{u}_n} = E\{\mathbf{u}_n\} = \int_{-\infty}^{\infty} u f_{\mathbf{u}_n}(u, n) du, \tag{60}$$

where E denotes and operator called mathematical expectation. The variance of $\mathbf{u}_n$ is defined as

$$\sigma_u^2[n] = \sigma_{\mathbf{u}_n}^2 = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})^2\} = \int_{-\infty}^{\infty} (u - m_{\mathbf{u}_n})^2 f_{\mathbf{u}_n}(u, n) du. \tag{61}$$

The autocorrelation of $\mathbf{u}_n$ is defined as

$$R_{uu}[n, m] = E\{\mathbf{u}_n \mathbf{u}_m^*\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_n u_m^* f_{\mathbf{u}_n,\mathbf{u}_m}(u_n, n, u_m, m) du_n du_m, \tag{62}$$

whereas the autocovariance sequence is defined as

$$C_{uu}[n, m] = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})(\mathbf{u}_m - m_{\mathbf{u}_m})^*\} = R_{uu}[n, m] - m_{\mathbf{u}_n} m_{\mathbf{u}_m}^*. \tag{63}$$

In the same way, given two stochastic processes $\mathbf{u}_n$ and $\mathbf{v}_n$ we can define the cross-correlation as

$$R_{uv}[n, m] = E\{\mathbf{u}_n \mathbf{v}_m^*\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_n v_m^* f_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) du_n dv_m, \tag{64}$$

whereas the cross-covariance sequence is defined as

$$C_{uv}[n,m] = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})(\mathbf{v}_m - m_{\mathbf{v}_m})^*\} = R_{uv}[n,m] - m_{\mathbf{u}_n} m^*_{\mathbf{v}_m}. \tag{65}$$

In general the statistical properties of a random variable may depend on n. For a stationary process the statistical properties are invariant to a shift of time origin. This means that the first order averages such as the mean and variance are independent of time and the seconde order averages such as the autocorrelation are dependent on the time diference. Thus, for a stationary process we can write

$$m_u[n] = m_u = E\{\mathbf{u}_n\} \tag{66}$$

$$\sigma_u^2[n] = \sigma_u^2 = E\{(\mathbf{u}_n - m_{\mathbf{u}})^2\} \tag{67}$$

$$R_{uu}[n+m,n] = R_{uu}[m] = E\{\mathbf{u}_{n+m}\mathbf{u}_n^*\}. \tag{68}$$

In many cases, the random processes are not stationary in the *strict sense* because their probability distributions are not time invariant but eqs. (66)–(68) still hold. We name those random processes as *wide-sense* stationary.

For a stationary random process, the essential characteristics of the process are represented by averages such as the mean, variance or autocorrelation. Therefore, it is essential to be able to estimate these quantities from finite-length segments of data. An estimator for the mean value is the *sample mean*, defined as

$$\hat{m}_u = \frac{1}{N}\sum_{n=0}^{N-1} u[n], \tag{69}$$

which is unbiased ($E\{\hat{m}_u\} = m_u$). An estimator for the variance is the *sample variance*, defined as

$$\hat{\sigma}_u^2 = \frac{1}{N}\sum_{n=0}^{N-1}(u[n] - \hat{m}_u)^2, \tag{70}$$

which is asymptotically unbiased ($\lim_{N\to\infty} E\{\hat{\sigma}_u^2\} = \sigma_u^2$). The estimators for the auto-covariance and cross-covariance are respectively defined as

$$\hat{C}_{uu}(\tau) = \frac{1}{N}\sum_{n=0}^{N-1}(u[n] - \hat{m}_u)(u[n-\tau] - \hat{m}_u), \tag{71}$$

$$\hat{C}_{uv}(\tau) = \frac{1}{N}\sum_{n=0}^{N-1}(u[n] - \hat{m}_u)(v[n-\tau] - \hat{m}_v). \tag{72}$$

Both estimators are asymptotically unbiased ($\lim_{N\to\infty} E\{\hat{C}_{uu}(\tau)\} = C_{uu}(\tau)$, $\lim_{N\to\infty} E\{\hat{C}_{uv}(\tau)\} = C_{uv}(\tau)$) and in addition it can be showed that

$$E\{\hat{C}(\tau)\} = \frac{N - |\tau|}{N} C(\tau). \tag{73}$$

While stochastic signals are not absolutely summable or square summable and consequently do not have Fourier Transforms, many of the properties of such signals can be summarized in terms of the autocorrelation or autocovariance sequence, for which the Fourier Transform often exists.

We define the *Power Spectrum Density* (PSD) as the Fourier Transform of the auto-covariance sequence

$$\Phi_{uu}(\omega) = \Phi_{uu}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} C_{uu}[n]e^{-j\omega n}, \tag{74}$$

and the *Cross Spectrum Density* (CSD) as the Fourier Transform of the cross-covariance sequence

$$\Phi_{uv}(\omega) = \Phi_{uv}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} C_{uv}[n]e^{-j\omega n}. \tag{75}$$

By definition of the auto-covariance and the Inverse Fourier Transform we can write

$$\sigma_u^2 = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})^2\} = C_{uu}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{uu}(\omega)d\omega. \tag{76}$$

It is possible to show that

$$\hat{\Phi}_{uu}(\omega) = \sum_{n=-\infty}^{\infty} \hat{C}_{uu}[n]e^{-j\omega n} = \frac{1}{N}|U(\omega)|^2 = P_{uu}(\omega), \tag{77}$$

and

$$\hat{\Phi}_{uv}(\omega) = \sum_{n=-\infty}^{\infty} \hat{C}_{uv}[n]e^{-j\omega n} = \frac{1}{N}U(\omega)V^*(\omega), \tag{78}$$

where $U(\omega)$ and $V(\omega)$ are the Discrete Fourier Transforms of $u[n]$ and $v[n]$ respectively and $P_{uu}(\omega)$ is the *Periodogram* of the sequence $u[n]$. Assuming that the sequence $u[n]$ is the sampled version of a stationary random signal $s(t)$ whose PSD $\Phi_{ss}(\Omega)$ is bandlimited by the antialiasing lowpass filter ($-2\pi\frac{f_s}{2} < \Omega < 2\pi\frac{f_s}{2}$), its PSD $\Phi_{uu}(\omega)$ is proportional to $\Phi_{ss}(\Omega)$ over the bandwith of the antialiasing filter, i.e.,

$$\Phi_{uu}(\omega) = \frac{1}{T_s}\Phi_{ss}\left(\frac{\omega}{T_s}\right), \quad |\omega| < \pi \Rightarrow \Phi_{uu}(f) = \Phi_{ss}\left(\frac{\omega}{T_s}\right) = \frac{\Phi_{uu}(\omega)}{f_s}, \quad |\omega| < \pi, f = \frac{\omega f_s}{2\pi}. \tag{79}$$

For a linear time-invariant (LTI) system with impulse response $h[n]$, we know that the output sequence $y[n]$ is related to the input sequence $u[n]$ through the convolution sum,

$$y[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k]. \tag{80}$$

We assume for convenience that $m_u = 0$. Then we have

$$m_y = \sum_{k=-\infty}^{\infty} h[k]m_u = 0. \tag{81}$$

The autocorrelation sequence for the output $y[n]$ is given by

$$\begin{aligned} R_{yy}[\tau] = E\{y[n]y[n-\tau]\} &= E\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]u[n-k]h[r]u[n-\tau-r]\right\} && (82)\\ &= \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]E\{u[n-k]u[n-\tau-r]\} && (83)\\ &= \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]R_{uu}(\tau+r-k] && (84)\end{aligned}$$

By making the substitution $-l = r - k$ we can write

$$R_{yy}[\tau] = \sum_{l=-\infty}^{\infty} R_{uu}(\tau - l) \sum_{r=-\infty}^{\infty} h[l+r]h[r] \tag{85}$$

$$R_{yy}[\tau] = \sum_{l=-\infty}^{\infty} R_{uu}(\tau - l) R_{hh}(l). \tag{86}$$

Taking into account that the Fourier Tranform of $R_{hh}(l) = h[n] * h[-n]$ is equal to $H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2$ and applying Fourier Transform to the last equation we can obtain the relationship

$$\Phi_{yy}(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 \Phi_{uu}(e^{j\omega}). \tag{87}$$

The cross-correlation between the input $u[n]$ and output $y[n]$ is given by

$$\begin{aligned} R_{uy}[\tau] = E\{u[n]y[n-\tau]\} &= E\left\{u[n] \sum_{k=-\infty}^{\infty} h[k]u[n-\tau-k]\right\} && (88)\\ &= \sum_{k=-\infty}^{\infty} h[k]E\{u[n]u[n-\tau-k]\} && (89)\\ &= \sum_{k=-\infty}^{\infty} h[k]R_{uu}(\tau+k] && (90)\\ &= h[\tau] * R_{uu}[-\tau] && (91)\\ &= h[\tau] * R_{uu}[\tau] && (92) \end{aligned}$$

Applying Fourier Transform to the last equation we can obtain the relationship

$$\Phi_{uy}(e^{j\omega}) = H(e^{j\omega})\Phi_{uu}(e^{j\omega}). \tag{93}$$

10 Frequency Response by sinusoidal excitation

Let us have a Discrete-Time LTI system described by

$$y[n] = G(z)u[n] + v[n] \tag{94}$$

where $v[n]$ represents a noise sequence. If the input sequence $u[n]$ is given by

$$u[n] = A\cos(\omega_o n) \tag{95}$$

we already showed (recall that $\cos(\omega_o n) = (e^{j\omega_o n} + e^{-j\omega_o n})/2$) that the output will be

$$y[n] = A\left|G(e^{j\omega_o})\right| \cos(\omega_o n + \arg[G(e^{j\omega_o})]) + v[n]. \tag{96}$$

Given the sums

$$I_c(N) = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\cos(\omega_o n), \quad I_s(N) = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\sin(\omega_o n), \tag{97}$$

we can insert (96) in (97) to obtain

$$I_c(N) = \frac{1}{N}\sum_{n=0}^{N-1} A\left|G(e^{j\omega_o})\right|\cos(\omega_o n + \arg[G(e^{j\omega_o})])\cos(\omega_o n) + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\cos(\omega_o n) \quad (98)$$

$$I_c(N) = A\left|G(e^{j\omega_o})\right|\frac{1}{2}\frac{1}{N}\sum_{n=0}^{N-1}\left[\cos(2\omega_o n + \arg[G(e^{j\omega_o})]) + \cos(\arg[G(e^{j\omega_o})])\right] + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\cos(\omega_o n) \quad (99)$$

$$I_c(N) = \frac{A}{2}\left|G(e^{j\omega_o})\right|\cos(\arg[G(e^{j\omega_o})]) + \frac{A}{2}\left|G(e^{j\omega_o})\right|\frac{1}{N}\sum_{n=0}^{N-1}\cos(2\omega_o n + \arg[G(e^{j\omega_o})]) + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\cos(\omega_o n) \quad (100)$$

and similarly

$$I_s(N) = -\frac{A}{2}\left|G(e^{j\omega_o})\right|\sin(\arg[G(e^{j\omega_o})]) + \frac{A}{2}\left|G(e^{j\omega_o})\right|\frac{1}{N}\sum_{n=0}^{N-1}\sin(2\omega_o n + \arg[G(e^{j\omega_o})]) + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\sin(\omega_o n). \quad (101)$$

When $N \to \infty$, the second and third terms in the expressions for $I_c(N)$ and $I_s(N)$ tend to zero. We finally obtain

$$I_c(N) = \frac{A}{2}\left|G(e^{j\omega_o})\right|\cos(\arg[G(e^{j\omega_o})]) \quad (102)$$

$$I_s(N) = -\frac{A}{2}\left|G(e^{j\omega_o})\right|\sin(\arg[G(e^{j\omega_o})]). \quad (103)$$

These two expressions suggest the following estimators for the Frequency Response:

$$\left|\hat{G}(e^{j\omega_o})\right| = \frac{\sqrt{I_c(N)^2 + I_s(N)^2}}{A/2} \quad (104)$$

$$\hat{\arg}[G(e^{j\omega_o})] = -\arctan\frac{I_s(N)}{I_c(N)}. \quad (105)$$

The Discrete Fourier Transform of the output sequence $y[n]$ is given by

$$Y(\omega) = \sum_{n=0}^{N-1} y[n]e^{-j\omega n}, \quad \omega = \frac{2\pi}{N}k\,(0 \le k \le N-1; 0 \le \omega < 2\pi). \quad (106)$$

Comparing this expression with (97) we can write

$$I_c(N) - jI_s(N) = \frac{1}{N}Y(\omega_o). \quad (107)$$

The Discrete Fourier Transform of the input sequence $u[n]$ is computed as

$$U(\omega) = \sum_{n=0}^{N-1} u[n]e^{-j\omega n} = \sum_{n=0}^{N-1} A\frac{e^{j\omega_o n} + e^{-j\omega_o n}}{2}e^{-j\omega n},\ \omega = \frac{2\pi}{N}k\,(0 \le k \le N-1; 0 \le \omega < 2\pi), \quad (108)$$

resulting

$$U(\omega) = \begin{cases} N\frac{A}{2} & \text{for } \omega = \omega_o \text{ if } \omega_o = \frac{2\pi}{N}k \text{ for some integer } 0 \leq k \leq N-1, \\ 0 & \text{otherwise.} \end{cases} \tag{109}$$

It is straightforward now to show that

$$\left|\hat{G}(e^{j\omega_o})\right| = \left|\frac{Y(\omega_o)}{U(\omega_o)}\right| \tag{110}$$

$$\hat{\arg}[G(e^{j\omega_o})] = \arg\left[\frac{Y(\omega_o)}{U(\omega_o)}\right]. \tag{111}$$

which means that an estimation of the Frequency Response at the frequency of the input signal can be computed based on the Discrete Fourier Transform of the input and output sequences:

$$\hat{G}(e^{j\omega_o}) = \frac{Y(\omega_o)}{U(\omega_o)}. \tag{112}$$

In addition to this estimator, eq. (93) suggests that the Frequency Response at the frequency of the input signal can also be estimated as:

$$\hat{\hat{G}}(e^{j\omega_o}) = \frac{\hat{\Phi}_{uy}(e^{j\omega_o})}{\hat{\Phi}_{uu}(e^{j\omega_o})}, \tag{113}$$

which reduces to (112) when we take into account (77) and (78).

References

[1] Alan V. Oppenheim and Ronald W. Schafer, "Discrete-Time Signal Processing," *Prentice Hall*, Second Edition, 1999.

[2] Lennart Ljung, "System Identification: Theory for the User," *Prentice Hall*, Second Edition, 1999.