System Identification and Robust Control

Lecture 8: Uncertainty in MIMO Systems

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General control problem formulation [3.8]

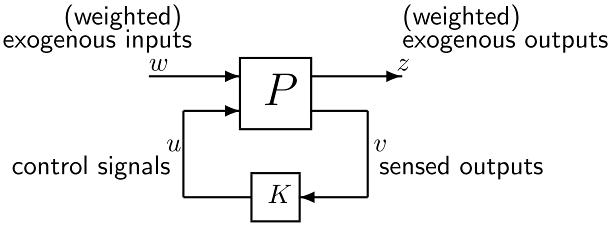


Figure 1: General control configuration for the case with no model uncertainty

- The overall control objective is to minimize some norm of the transfer function from w to z, for example, the $\mathcal{H}_\infty$ norm. The controller design problem is then:
  - Find a controller K which based on the information in v, generates a control signal u which counteracts the influence of w on z, thereby minimizing the closed-loop norm from w to z.

Obtaining the generalized plant P [3.8.1]

- Almost any linear control problem can be formulated using the block diagram in Fig. 1
- The routines in MATLAB for synthesizing $\mathcal{H}_\infty$ optimal controllers assume that the problem is in the general form of Figure 1.

Example: One degree-of-freedom feedback control configuration.

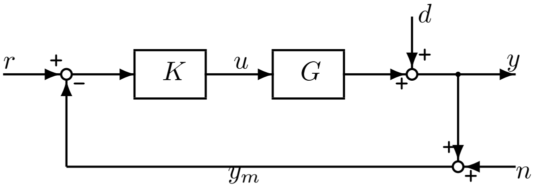


Figure 2: One degree-of-freedom control configuration

Equivalent representation of Figure 2 where the error signal to be minimized is $z = y - r$ and the input to the controller is $v = r - y_m$.

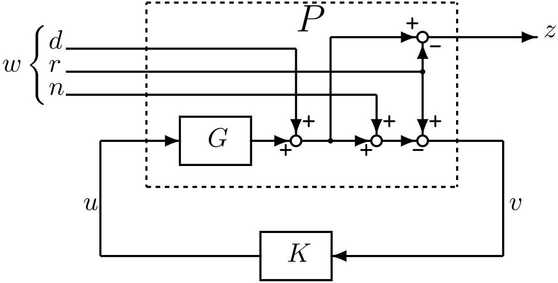


Figure 3: General control configuration equivalent to Figure 2

To construct P one should note that it is an open-loop system and remember to break all "loops" entering and exiting the controller K .

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; z = e = y - r; v = r - y_m = r - y - n \qquad (1)$$

$$\begin{aligned} z &= y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu \\ v &= r - y_m = r - Gu - d - n = \\ &= -Iw_1 + Iw_2 - Iw_3 - Gu \end{aligned}$$

P which represents the transfer function from $\begin{bmatrix} w & u \end{bmatrix}^T$ to $\begin{bmatrix} z & v \end{bmatrix}^T$ is

$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \qquad (2)$$

Note 1: P does *not* contain the controller!
Note 2: Alternatively, P can be obtained by inspection from Figure 3.

Remark. In MATLAB we may obtain P via `simulink`, or we may use the `sysic` program in the μ-toolbox. The code in Table 1 generates the generalized plant P in (2) for Figure 2.

Table 1: **MATLAB program to generate P**

```
% Uses the Mu-toolbox
systemnames = 'G';                        % G is the SISO plant.
inputvar = '[d(1);r(1);n(1);u(1)]';   % Consists of vectors w and u.
input_to_G = '[u]';
outputvar = '[G+d-r; r-G-d-n]';         % Consists of vectors z and v.
sysoutname = 'P';
sysic;
```

Including weights in P [3.8.2]

To get a meaningful controller synthesis problem, for example, in terms of the $\mathcal{H}_\infty$ norm, we generally have to include weights W_z and W_w in the generalized plant P, see Figure 4.

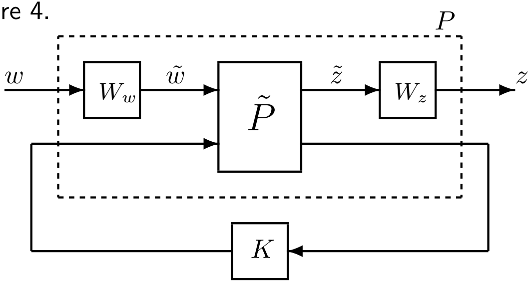


Figure 4: General control configuration for the case with no model uncertainty

That is, we consider the weighted or normalized exogenous inputs w, and the weighted or normalized controlled outputs $z = W_z \tilde{z}$. The weighting matrices are usually frequency dependent and typically selected such that weighted signals w and z are of magnitude 1, that is, the norm from w to z should be less than 1.

Example: Stacked $S/T/KS$ problem.

Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem

$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix} \tag{3}$$

where K is a stabilizing controller. In other words, we have $z = Nw$ and the objective is to minimize the $\mathcal{H}_\infty$ norm from w to z.

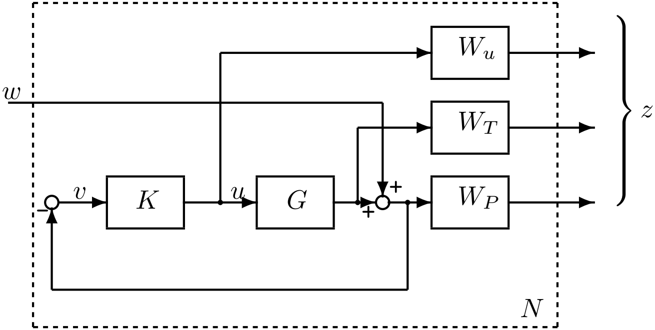


Figure 5: Block diagram corresponding to generalized plant in (3)

$$\begin{aligned} z_1 &= W_u u \\ z_2 &= W_T G u \\ z_3 &= W_P w + W_P G u \\ v &= -w - G u \end{aligned}$$

The generalized plant P from $\begin{bmatrix} w & u \end{bmatrix}^T$ to $\begin{bmatrix} z & v \end{bmatrix}^T$ is

$$P = \left[\begin{array}{c|c} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ \hline -I & -G \end{array}\right] \tag{4}$$

Partitioning the generalized plant P [3.8.3]

We often partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad (5)$$

so that

$$z = P_{11}w + P_{12}u \quad (6)$$

$$v = P_{21}w + P_{22}u \quad (7)$$

In Example "Stacked $S/T/KS$ problem" we get from (4)

$$P_{11} = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} \tag{8}$$

$$P_{21} = -I, \quad P_{22} = -G \tag{9}$$

Note that P_{22} has dimensions compatible with the controller K in Figure 4.

Analysis: Closing the loop to get N [3.8.4]

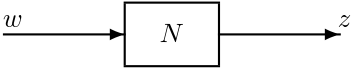


Figure 6: General block diagram for analysis with no uncertainty

For *analysis* of closed-loop performance we may absorb K into the interconnection structure and obtain the system N as shown in Figure 6 where

$$z = Nw \tag{10}$$

To find N, which is a function of K, we first partition the generalized plant P as given in (5)-(7), combine this with the controller equation

$$u = Kv, \tag{11}$$

and eliminate u and v from equations (6), (7) and (11) to yield $z = Nw$ where

$$N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \stackrel{\Delta}{=} F_l(P, K) \tag{12}$$

Here $F_l(P, K)$ denotes a lower *linear fractional transformation (LFT)* of P with K as the parameter. In words, N is obtained from Figure 1 by using K to close a lower feedback loop around P. Since positive feedback is used in the general configuration in Figure 1 the term $(I - P_{22}K)^{-1}$ has a negative sign.

Example: We want to derive N for the partitioned P in (8) and (9) using the LFT-formula in (12). We get

$$N = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix} + \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} K(I+GK)^{-1}(-I) = \begin{bmatrix} -W_u KS \\ -W_T T \\ W_P S \end{bmatrix}$$

where we have made use of the identities $S = (I+GK)^{-1}$, $T = GKS$ and $I - T = S$.

In the MATLAB μ-Toolbox we can evaluate $N = F_l(P,K)$ using the command `N=starp(P,K)`. Here `starp` denotes the matrix star product which generalizes the use of LFTs.

Further examples [3.8.5]

Example: Consider the control system in Figure 7, where y_1 is the output we want to control, y_2 is a secondary output (extra measurement), and we also measure the disturbance d. The control configuration includes a two degrees-of-freedom controller, a feedforward controller and a local feedback controller based on the extra measurement y_2.

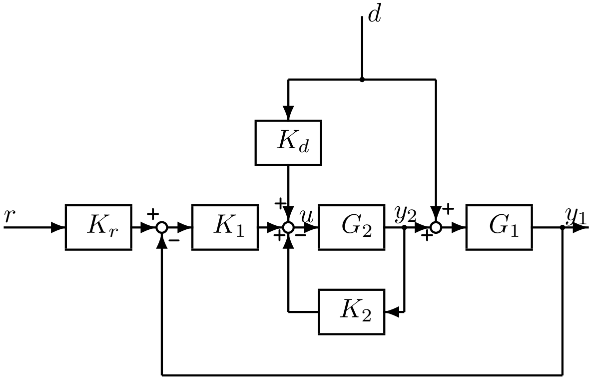


Figure 7: System with feedforward, local feedback and two degrees-of-freedom control

To recast this into our standard configuration of Figure 1 we define

$$w = \begin{bmatrix} d \\ r \end{bmatrix}; \quad z = y_1 - r; \quad v = \begin{bmatrix} r \\ y_1 \\ y_2 \\ d \end{bmatrix} \tag{13}$$

$$K = \begin{bmatrix} K_1 K_r & -K_1 & -K_2 & K_d \end{bmatrix} \tag{14}$$

We get

$$P = \left[\begin{array}{cc|c} G_1 & -I & G_1 G_2 \\ \hline 0 & I & 0 \\ G_1 & 0 & G_1 G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{array} \right] \tag{15}$$

Then partitioning P as in (6) and (7) yields:
$P_{22} = \begin{bmatrix} 0^T & (G_1 G_2)^T & G_2^T & 0^T \end{bmatrix}^T$.

Deriving P from N [3.8.6]

For cases where N is given and we wish to find a P such that

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

it is usually best to work from a block diagram representation. This was illustrated above for the stacked N in (3). Alternatively, the following procedure may be useful:

1. Set $K = 0$ in N to obtain P_{11}.
2. Define $Q = N - P_{11}$ and rewrite Q such that each term has a common factor $R = K(I - P_{22}K)^{-1}$ (this gives P_{22}).
3. Since $Q = P_{12}RP_{21}$, we can now usually obtain P_{12} and P_{21} by inspection.

Example: Weighted sensitivity. We will use the above procedure to derive P when

$$N = w_P S = w_P(I + GK)^{-1}$$

, where w_P is a scalar weight.

1. $P_{11} = N(K = 0) = w_P I$.
2. $Q = N - w_P I = w_P(S - I) = -w_P T = -w_P GK(I + GK)^{-1}$, and we have $R = K(I + GK)^{-1}$ so $P_{22} = -G$.
3. $Q = -w_P GR$ so we have $P_{12} = -w_P G$ and $P_{21} = I$, and we get

$$P = \begin{bmatrix} w_P I & -w_P G \\ I & -G \end{bmatrix} \tag{16}$$

General control configuration with model uncertainty [3.8.8]

The general control configuration in Figure 1 may be extended to include model uncertainty. Here the matrix Δ is a *block-diagonal* matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$.

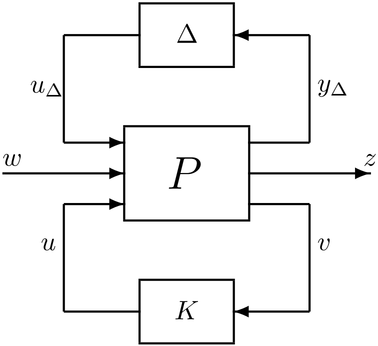


Figure 8: General control configuration for the case with model uncertainty

Introduction to Multivariable Control

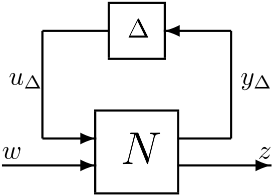


Figure 9: General block diagram for analysis with uncertainty included

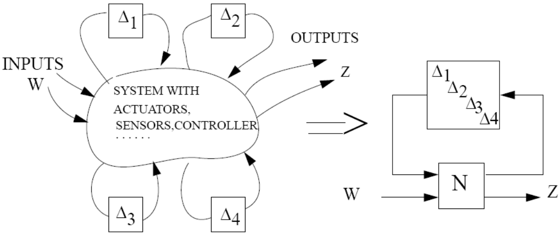


Figure 10: Rearranging a system with multiple perturbations into the $N\Delta$-structure

The block diagram in Figure 8 in terms of P (for synthesis) may be transformed into the block diagram in Figure 9 in terms of N (for analysis) by using K to close a lower loop around P. The same *lower LFT* as found in (12) applies, and

$$N = F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{17}$$

To evaluate the perturbed (uncertain) transfer function from external inputs w to external outputs z, we use Δ to close the upper loop around N (see Figure 9), resulting in an *upper LFT*:

$$z = F_u(N,\Delta)w; \; F_u(N,\Delta) \stackrel{\Delta}{=} N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{18}$$

Remark 1 Almost any control problem with uncertainty can be represented by Figure 8. First represent each source of uncertainty by a perturbation block, Δ_i, which is normalized such that $\|\Delta_i\| \leq 1$. Then “pull out” each of these blocks from the system so that an input and an output can be associated with each Δ_i as shown in Figure 10(a). Finally, collect these perturbation blocks into a large block-diagonal matrix having perturbation inputs and outputs as shown in Figure 10(b).

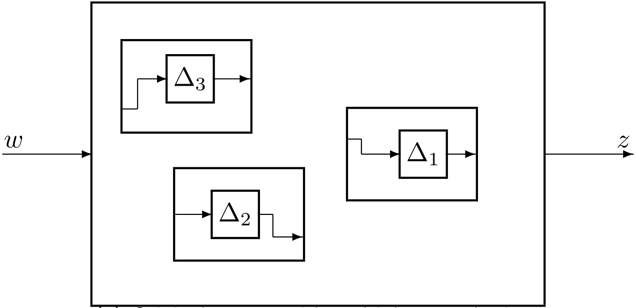
General configuration with uncertainty [8.1]

For our robustness analysis we use a system representation in which the uncertain perturbations are "pulled out" into a block-diagonal matrix,

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix} \tag{19}$$

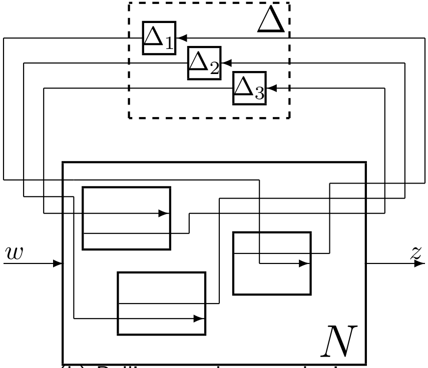
where each Δ_i represents a specific source of uncertainty.

Uncertainty in MIMO Systems



(a) Original system with multiple perturbations

Uncertainty in MIMO Systems



(b) Pulling out the perturbations

Figure 11: Rearranging an uncertain system into the $N\Delta$-structure

Uncertainty in MIMO Systems

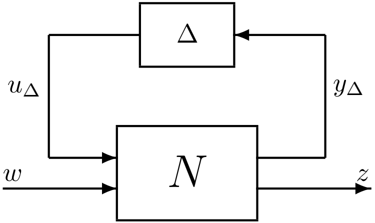


Figure 12: $N\Delta$-structure for robust performance analysis

If we also pull out the controller K, we get the generalized plant P, as shown in Figure 13. *For analysis* of the uncertain system, we use the $N\Delta$-structure in Figure 12.

Uncertainty in MIMO Systems

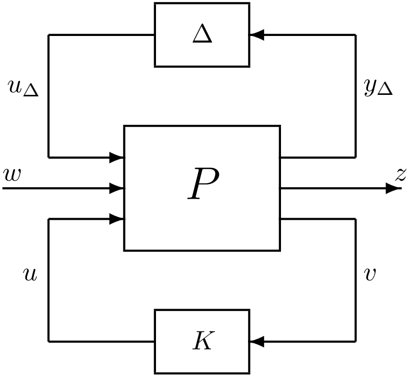


Figure 13: General control configuration (for controller synthesis)

Uncertainty in MIMO Systems

Consider Figure 11 where it is shown how to pull out the perturbation blocks to form Δ and the nominal system N. N is related to P and K by a lower LFT

$$N = F_l(P, K) \stackrel{\Delta}{=} P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{20}$$

Similarly, the uncertain closed-loop transfer function from w to z, $z = Fw$, is related to N and Δ by an upper LFT,

$$F = F_u(N, \Delta) \stackrel{\Delta}{=} N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{21}$$

To analyze robust stability of F, we can then rearrange the system into the $M\Delta$-structure of Figure 14 where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.

Uncertainty in MIMO Systems

To analyze robust stability of F, we can then arrange the system into the $M\Delta$-structure of Figure 14, where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.

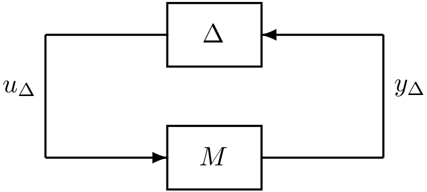


Figure 14: $M\Delta$-structure for robust stability analysis

Representing uncertainty [8.2]

As usual, each individual perturbation is assumed to be stable and is normalized,

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \;\forall\omega \tag{22}$$

For a complex scalar perturbation we have $|\delta_i(j\omega)| \leq 1, \;\forall\omega$, and for a real scalar perturbation $-1 \leq \delta_i \leq 1$. Since the maximum singular value of a block diagonal matrix is equal to the largest of the maximum singular values of the individual blocks, it then follows for $\Delta = \mathrm{diag}\{\Delta_i\}$ that

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \;\forall\omega, \;\forall i \quad \Leftrightarrow \quad \boxed{\|\Delta\|_\infty \leq 1} \tag{23}$$

Note that Δ has *structure*, and therefore in the robustness analysis we do *not* want to allow all Δ s.t. (23) is satisfied. Only the subset which has the block-diagonal structure in (19) should be considered. In some cases the blocks in Δ may be repeated or may be real, that is, we have additional structure. For example, repetition is often needed to handle parametric uncertainty (see Section 7.7.3 in the book).

Parametric uncertainty [8.2.2]

The representation of parametric uncertainty discussed for SISO systems carries straightforward over to MIMO systems.

Unstructured uncertainty [8.2.3]

We define *unstructured* uncertainty as the use of a "full" complex perturbation matrix Δ, usually with dimensions compatible with those of the plant, where at each frequency any $\Delta(j\omega)$ satisfying $\bar{\sigma}(\Delta(j\omega)) \leq 1$ is allowed.

Six common forms of unstructured uncertainty are shown in Figure 15. In Figure 15(a), (b) and (c) are shown three *feedforward* forms; additive uncertainty, multiplicative input uncertainty and multiplicative output uncertainty:

$$\Pi_A : \qquad G_p = G + E_A; \qquad E_a = w_A \Delta_a \tag{24}$$

$$\Pi_I : \qquad G_p = G(I + E_I); \qquad E_I = w_I \Delta_I \tag{25}$$

$$\Pi_O : \qquad G_p = (I + E_O)G; \qquad E_O = w_O \Delta_O \tag{26}$$

Uncertainty in MIMO Systems

In Figure 15(d), (e) and (f) are shown three *feedback* or *inverse* forms; inverse additive uncertainty, inverse multiplicative input uncertainty and inverse multiplicative output uncertainty:

$$\Pi_{iA}: \qquad G_p = G(I - E_{iA}G)^{-1}; \qquad E_{iA} = w_{iA}\Delta_{iA} \tag{27}$$

$$\Pi_{iI}: \qquad G_p = G(I - E_{iI})^{-1}; \qquad E_{iI} = w_{iI}\Delta_{iI} \tag{28}$$

$$\Pi_{iO}: \qquad G_p = (I - E_{iO})^{-1}G; \qquad E_{iO} = w_{iO}\Delta_{iO} \tag{29}$$

The negative sign in front of the E's does not really matter here since we assume that Δ can have any sign. Δ denotes the normalized perturbation and E the "actual" perturbation. We have here used scalar weights w, so $E = w\Delta = \Delta w$, but sometimes one may want to use matrix weights, $E = W_2 \Delta W_1$ where W_1 and W_2 are given transfer function matrices.

Uncertainty in MIMO Systems

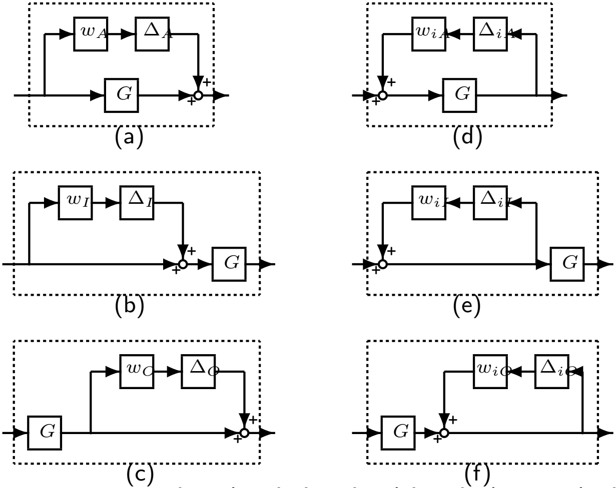


Figure 15: Six common uncertainty descriptions involving single perturbations; (a) Additive uncertainty, (b) Multiplicative input uncertainty, (c) Multiplicative output uncertainty, (d) Inverse additive uncertainty, (e) Inverse multiplicative input uncertainty, (f) Inverse multiplicative output uncertainty

Obtaining P, N and M [8.3]

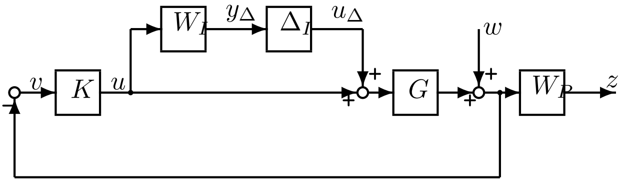


Figure 16: System with multiplicative input uncertainty and performance measured at the output

Example 1: System with input uncertainty (Figure 16). We want to derive the generalized plant P in Figure 13 which has inputs $\begin{bmatrix} u_\Delta & w & u \end{bmatrix}^T$ and outputs $\begin{bmatrix} y_\Delta & z & v \end{bmatrix}^T$. By writing down the equations or simply by inspecting Figure 16 (remember to remove K and Δ_I) we get

$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix} \tag{30}$$

Next, we want to derive the matrix N corresponding to Figure 12. First, partition P to be compatible with K, i.e.

$$P_{11} = \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix}, \; P_{12} = \begin{bmatrix} W_I \\ W_P G \end{bmatrix} \tag{31}$$

$$P_{21} = \begin{bmatrix} -G & -I \end{bmatrix}, \; P_{22} = -G \tag{32}$$

and then find $N = F_l(P, K)$ using (20).

We get

$$N = \begin{bmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix} \quad (33)$$

Alternatively, we can derive N directly from Figure 16 by evaluating the closed-loop transfer function from inputs $\begin{bmatrix} u_\Delta \\ w \end{bmatrix}$ to outputs $\begin{bmatrix} y_\Delta \\ z \end{bmatrix}$ (*without* breaking the loop before and after K).

For example, to derive N_{12}, which is the transfer function from w to y_Δ, we start at the output (y_Δ) and move backwards to the input (w) using the MIMO Rule (we first meet W_I, then $-K$ and we then exit the feedback loop and get the term $(I+GK)^{-1}$).

The upper left block, N_{11}, in (33) is the transfer function from u_Δ to y_Δ. This is the transfer function M needed in Figure 14 for evaluating robust stability. Thus, we have $M = -W_I KG(I+KG)^{-1} = -W_I T_I$.

Robust stability & performance [8.4]

1. *Robust stability (RS) analysis*: with a given controller K we determine whether the system remains stable for all plants in the uncertainty set.
2. *Robust performance (RP) analysis*: if RS is satisfied, we determine how "large" the transfer function from exogenous inputs w to outputs z may be for all plants in the uncertainty set.

In Figure 12, w represents the exogenous inputs (normalized disturbances and references), and z the exogenous outputs (normalized errors). We have $z = F(\Delta)w$, where from (21)

$$F = F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{34}$$

Uncertainty in MIMO Systems

We here use the $\mathcal{H}_\infty$ norm to define performance and require for RP that $\|F(\Delta)\|_\infty \leq 1$ for all allowed Δ's. A typical choice is $F = w_P S_p$ (the weighted sensitivity function), where w_P is the performance weight (capital P for performance) and S_p represents the set of perturbed sensitivity functions (lower-case p for perturbed).

Uncertainty in MIMO Systems

In terms of the $N\Delta$-structure in Figure 12 our requirements for stability and performance are

$$
\begin{aligned}
\text{NS} &\overset{\text{def}}{\Leftrightarrow} N \text{ is internally stable} && (35)\\
\text{NP} &\overset{\text{def}}{\Leftrightarrow} \|N_{22}\|_\infty < 1; \quad \text{and NS} && (36)\\
\text{RS} &\overset{\text{def}}{\Leftrightarrow} F = F_u(N,\Delta) \text{ is stable } \forall\Delta, \|\Delta\|_\infty \le 1; && (37)\\
&\qquad \text{and NS}\\
\text{RP} &\overset{\text{def}}{\Leftrightarrow} \|F\|_\infty < 1, \quad \forall\Delta, \|\Delta\|_\infty \le 1; && (38)\\
&\qquad \text{and NS}
\end{aligned}
$$

Robust stability of the $M\Delta$-structure [8.5]

Consider the uncertain $N\Delta$-system in Figure 12 for which the transfer function from w to z is, as in (34), given by

$$F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{39}$$

Suppose that the system is nominally stable (with $\Delta = 0$), that is, N is stable. We also assume that Δ is stable. Thus, when we have nominal stability (NS), the stability of the system in Figure 12 is equivalent to the stability of the $M\Delta$-structure in Figure 14 where $M = N_{11}$.

Uncertainty in MIMO Systems

Theorem

Determinant stability condition (Real or complex perturbations). *Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Consider the convex set of perturbations Δ, such that if Δ' is an allowed perturbation then so is $c\Delta'$ where c is any* <u>*real*</u> *scalar such that $|c| \leq 1$. Then the $M\Delta$-system in Figure 14 is stable for all allowed perturbations (we have RS) if and only if*

$$\text{Nyquist plot of } \det\left(I - M(s)\Delta(s)\right) \text{ does not encircle the origin, } \forall\Delta \tag{40}$$

$$\Leftrightarrow \quad \boxed{\det\left(I - M(j\omega)\Delta(j\omega)\right) \neq 0, \quad \forall\omega, \forall\Delta} \tag{41}$$

$$\Leftrightarrow \quad \lambda_i(M\Delta) \neq 1, \quad \forall i, \forall\omega, \forall\Delta \tag{42}$$

Uncertainty in MIMO Systems

Theorem

Generalized (MIMO) Nyquist theorem. *Let* P_{ol} *denote the number of open-loop unstable poles in* $L(s)$*. The closed-loop system with loop transfer function* $L(s)$ *and negative feedback is stable if and only if the Nyquist plot of* $\det(I + L(s))$
i) makes P_{ol} *anti-clockwise encirclements of the origin, and*
ii) does not pass through the origin.

Note: By "Nyquist plot of $\det(I + L(s))$" we mean "the image of $\det(I + L(s))$ as s goes clockwise around the Nyquist D-contour".

RS for unstructured uncertainty [8.6]

Theorem

RS for unstructured ("full") perturbations. *Assume that the nominal system $M(s)$ is stable (NS) and that the perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system in Figure 14 is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \leq 1$ (i.e. we have RS) if and only if*

$$\boxed{\bar{\sigma}(M(j\omega)) < 1 \quad \forall w} \qquad \Leftrightarrow \qquad \boxed{\|M\|_\infty < 1} \tag{43}$$

Uncertainty in MIMO Systems

Theorem

Small Gain Theorem. *Consider a system with a stable loop transfer function* $L(s)$*. Then the closed-loop system is stable if*

$$\|L(j\omega)\| < 1 \quad \forall\omega \tag{44}$$

where $\|L\|$ *denotes any matrix norm satisfying* $\|AB\| \leq \|A\| \cdot \|B\|$*, for example the singular value* $\bar{\sigma}(L)$*.*

RS with structured uncertainty [8.7]

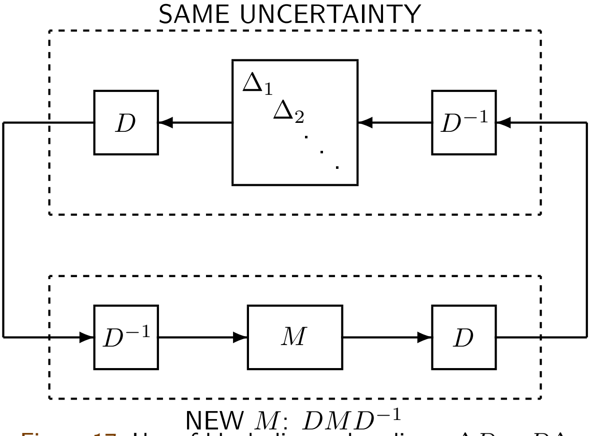


Figure 17: Use of block-diagonal scalings, $\Delta D = D\Delta$

Uncertainty in MIMO Systems

Consider now the presence of structured uncertainty, where $\Delta = \mathrm{diag}\{\Delta_i\}$ is block-diagonal. To test for robust stability we rearrange the system into the $M\Delta$-structure and we have from (43)

$$\mathrm{RS} \quad \text{if} \quad \bar{\sigma}(M(j\omega)) < 1, \forall\omega \tag{45}$$

We have here written "if" rather than "if and only if" since this condition is only necessary for RS when Δ has "no structure" (full-block uncertainty). To reduce conservativism introduce the block-diagonal scaling matrix

$$D = \mathrm{diag}\{d_i I_i\} \tag{46}$$

where d_i is a scalar and I_i is an identity matrix of the same dimension as the i'th perturbation block, Δ_i (Figure 17). This clearly has no effect on stability.

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(DMD^{-1}) < 1, \forall\omega \tag{47}$$

This applies for any D in (46), and therefore the "most improved" (least conservative) RS-condition is obtained by minimizing at each frequency the scaled singular value, and we have

$$\boxed{\text{RS} \quad \text{if} \quad \min_{D(\omega)\in\mathcal{D}} \bar{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1}) < 1, \forall\omega} \tag{48}$$

where $\mathcal{D}$ is the set of block-diagonal matrices whose structure is compatible to that of Δ, i.e, $\Delta D = D\Delta$.

The structured singular value [8.8]

The structured singular value (denoted Mu, mu, SSV or μ) is a function which provides a generalization of the singular value, $\bar{\sigma}$, and the spectral radius, ρ. We will use μ to get necessary and sufficient conditions for robust stability and also for robust performance. The name "structured singular value" is used because μ generalizes the singular value RS-condition, $\bar{\sigma}(M) \leq 1, \forall\omega$ in (43), to the case when Δ has structure (and also to cases where parts of Δ are real). How is μ defined? A simple statement is:

> *Find the smallest structured* Δ *(measured in terms of* $\bar{\sigma}(\Delta)$*) which makes* $\det(I - M\Delta) = 0$*; then* $\mu(M) = 1/\bar{\sigma}(\Delta)$*.*

Mathematically,

$$\mu(M)^{-1} \triangleq \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for structured } \Delta\} \quad (49)$$

Clearly, $\mu(M)$ depends not only on M but also on the allowed structure for Δ. This is sometimes shown explicitly by using the notation $\mu_\Delta(M)$.

Remark. For the case where Δ is "unstructured" (a full matrix), the smallest Δ which yields singularity has $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M)$, and we have $\mu(M) = \bar{\sigma}(M)$.

Definition

Structured Singular Value. Let M be a given complex matrix and let $\Delta = \mathrm{diag}\{\Delta_i\}$ denote a set of complex matrices with $\bar{\sigma}(\Delta) \leq 1$ and with a given block-diagonal structure (in which some of the blocks may be repeated and some may be restricted to be real). The real non-negative function $\mu(M)$, called the structured singular value, is defined by

$$\mu(M) \triangleq \frac{1}{\min\{k_m | \det(I - k_m M\Delta) = 0, \bar{\sigma}(\Delta) \leq 1\}} \quad (50)$$

If no such structured Δ exists then $\mu(M) = 0$.

A value of $\mu = 1$ means that there exists a perturbation with $\bar{\sigma}(\Delta) = 1$ which is just large enough to make $I - M\Delta$ singular. A larger value of μ is "bad" as it means that a smaller perturbation makes $I - M\Delta$ singular, whereas a smaller value of μ is "good".

RS - Structured uncertainty [8.9]

Consider stability of the $M\Delta$-structure in Figure 14 for the case where Δ is a set of norm-bounded block-diagonal perturbations. From the determinant stability condition:

$$\det(I - M(j\omega)\Delta(j\omega)) \neq 0, \quad \forall\omega, \forall\Delta, \bar{\sigma}(\Delta(j\omega)) \leq 1 \quad \forall\omega \quad (51)$$

This is just a "yes/no" condition. To find the factor k_m by which the system is robustly stable, we scale the uncertainty Δ by k_m, and look for the smallest k_m that yields "borderline instability," namely

$$\det(I - k_m M(j\omega)\Delta(j\omega)) = 0 \quad (52)$$

By definition, this value is $k_m = 1/\mu(M)$. We obtain the following necessary and sufficient condition for stability.

Theorem

RS for block-diagonal perturbations (real or complex). *Assume that the nominal system M and the perturbations Δ are stable. Then the $M\Delta$-system in Figure 14 is stable for all allowed perturbations with $\bar{\sigma}(\Delta) \leq 1, \forall \omega$, <u>if and only if</u>*

$$\boxed{\mu(M(j\omega)) < 1, \quad \forall \omega} \tag{53}$$

Condition (53) for robust stability may be rewritten as

$$\text{RS} \;\Leftrightarrow\; \mu(M(j\omega))\;\bar{\sigma}(\Delta(j\omega)) < 1, \quad \forall \omega \tag{54}$$

which may be interpreted as a "generalized small gain theorem" that also takes into account the *structure* of Δ.

Example: RS with diagonal input uncertainty
Consider robust stability of the feedback system in Figure 16 for the case when the multiplicative input uncertainty is diagonal. A nominal 2×2 plant and the controller (which represents PI-control of a distillation process using the DV-configuration) is given by

$$G(s) = \frac{1}{\tau s+1}\begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}; \qquad K(s) = \frac{1+\tau s}{s}\begin{bmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{bmatrix} \tag{55}$$

(time in minutes). The controller results in a nominally stable system with acceptable performance. Assume there is complex multiplicative uncertainty in *each* manipulated input of magnitude

$$w_I(s) = \frac{s + 0.2}{0.5s + 1} \tag{56}$$

On rearranging the block diagram to match the $M\Delta$-structure in Figure 14 we get $M = w_I KG(I + KG)^{-1} = w_I T_I$ (recall (33)), and the RS-condition $\mu(M) < 1$ in Theorem 6 yields

$$\text{RS} \Leftrightarrow \mu_{\Delta_I}(T_I) < \frac{1}{|w_I(j\omega)|} \quad \forall\omega, \quad \Delta_I = \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix} \tag{57}$$

This condition is shown graphically in Figure 18 so the system is robustly stable. Also in Figure 18, $\bar{\sigma}(T_I)$ can be seen to be larger than $1/|w_I(j\omega)|$ over a wide frequency range. This shows that the system would be unstable for full-block input uncertainty (Δ_I full).

Uncertainty in MIMO Systems

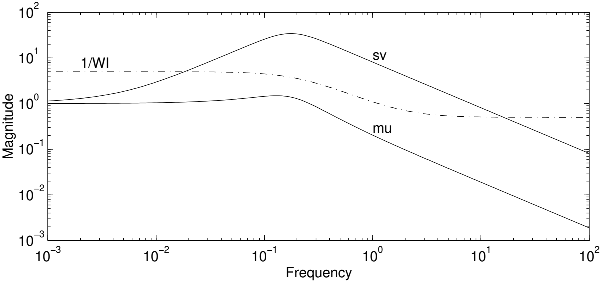


Figure 18: Robust stability for diagonal input uncertainty is guaranteed since $\mu_{\Delta_I}(T_I) < 1/|w_I|, \ \forall\omega$. The use of unstructured uncertainty and $\bar{\sigma}(T_I)$ is conservative

Robust performance [8.10]

- With an $\mathcal{H}_\infty$ performance objective, the RP-condition is identical to a RS-condition with an additional perturbation block.
- In Figure 19 step B is the key step.
- Δ_P (where capital P denotes Performance) is always a full matrix. It is a fictitious uncertainty block representing the $\mathcal{H}_\infty$ performance specification.

Testing RP using μ [8.10.1]

Theorem

Robust performance. *Rearrange the uncertain system into the $N\Delta$-structure of Figure 19. Assume nominal stability such that N is (internally) stable. Then*

$$\begin{aligned} \text{RP} \quad &\overset{\text{def}}{\Leftrightarrow} \quad \|F\|_\infty = \|F_u(N,\Delta)\|_\infty < 1, \quad \forall \|\Delta\|_\infty \le 1 \\ &= \quad \boxed{\mu_{\widehat{\Delta}}(N(j\omega)) < 1, \quad \forall w} \end{aligned} \tag{58}$$

where μ is computed with respect to the structure

$$\widehat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \tag{59}$$

and Δ_P is a full complex perturbation with the same dimensions as F^T.

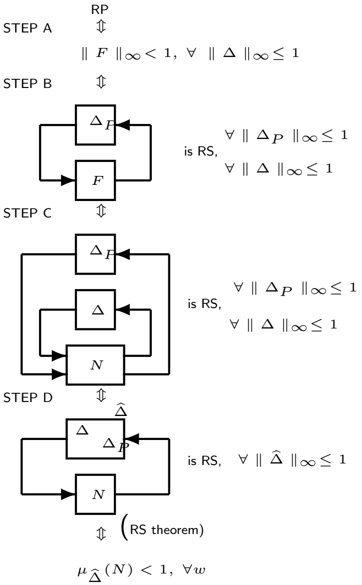


Figure 19: RP as structured RS. $F = F_u(N, \Delta)$

Summary of μ-conditions for NP, RS, RP [8.10.2]

Rearrange the uncertain system into the $N\Delta$- structure, where the block-diagonal perturbations satisfy $\|\Delta\|_\infty \leq 1$.
Introduce

$$F = F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

and let the performance requirement (RP) be $\|F\|_\infty \leq 1$ for all allowable perturbations. Then we have:

$$\begin{aligned}
\text{NS} &= N \text{ (internally) stable} && (60)\\
\text{NP} &= \bar{\sigma}(N_{22}) = \mu_{\Delta_P}(N_{22}) < 1,\ \forall\omega,\ \text{and NS} && (61)\\
\text{RS} &= \mu_{\Delta}(N_{11}) < 1,\ \forall\omega,\ \text{and NS} && (62)\\
\text{RP} &= \mu_{\tilde{\Delta}}(N) < 1,\ \forall\omega,\ \tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}, && (63)\\
&\quad \text{and NS}
\end{aligned}$$

μ-synthesis and DK-iteration [8.12]

The structured singular value μ is a very powerful tool for the analysis of robust performance with a given controller. However, one may also seek to find the controller that minimizes a given μ-condition: this is the μ-synthesis problem: $\min_K \mu(N)$

DK-iteration [8.12.1]

At present there is no direct method to synthesize a μ-optimal controller. However, for complex perturbations a method known as DK-iteration is available. It combines $\mathcal{H}_\infty$-synthesis and μ-analysis, and often yields good results. The starting point is the upper bound on μ in terms of the scaled singular value

$$\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

Uncertainty in MIMO Systems

The idea is to find the controller that minimizes the peak value over frequency of this upper bound, namely

$$\min_K \left(\min_{D \in \mathcal{D}} \|DN(K)D^{-1}\|_\infty \right) \quad (64)$$

by alternating between minimizing $\|DN(K)D^{-1}\|_\infty$ with respect to either K or D (while holding the other fixed).

1. **K-step.** Synthesize an $\mathcal{H}_\infty$ controller for the scaled problem, $\min_K \|DN(K)D^{-1}\|_\infty$ with fixed $D(s)$.
2. **D-step.** Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

Controller Design [9]

Trade-offs in MIMO feedback design [9.1]

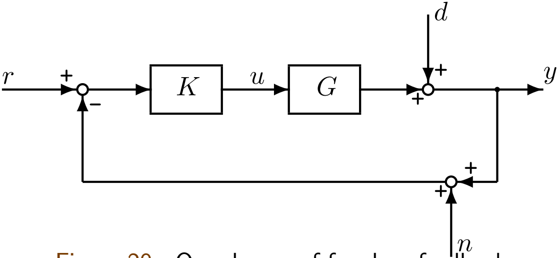


Figure 20: One degree-of-freedom feedback

$$y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s) \quad (65)$$

$$u(s) = K(s)S(s)\left[r(s) - n(s) - d(s)\right] \quad (66)$$

Controller Design

Closed-loop objectives:

1. For *disturbance rejection* make $\bar{\sigma}(S)$ small.

2. For *noise attenuation* make $\bar{\sigma}(T)$ small.

3. For *reference tracking* make $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$.

4. For *control energy reduction* make $\bar{\sigma}(KS)$ small.

5. For *robust stability* in the presence of an additive perturbation make $\bar{\sigma}(KS)$ small.

6. For *robust stability* in the presence of a multiplicative output perturbation make $\bar{\sigma}(T)$ small.

The closed-loop requirements 1 to 6 cannot all be satisfied simultaneously. Feedback design is therefore a trade-off over frequency of conflicting objectives.

Controller Design

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \qquad (67)$$

- At frequencies where $\underline{\sigma}(L) >> 1$, we have $\bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$
- At frequencies where $\bar{\sigma}(L) << 1$, we have $\bar{\sigma}(T) \approx \bar{\sigma}(L)$
- At the bandwidth frequency $(1/\bar{\sigma}(S(j\omega_B)) = \sqrt{2} = 1/41)$, we have $0.41 \leq \underline{\sigma}(L(j\omega_B)) \leq 2.41$

Controller Design

Over specified frequency ranges, we can approximate the closed-loop requirements by the following open-loop objectives:

1. For *disturbance rejection* make $\underline{\sigma}(GK)$ large; valid for frequencies at which $\underline{\sigma}(GK) \gg 1$.
2. For *noise attenuation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.
3. For *reference tracking* make $\underline{\sigma}(GK)$ large; valid for frequencies at which $\underline{\sigma}(GK) \gg 1$.
4. For *control energy reduction* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.
5. For *robust stability to an additive perturbation* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.
6. For *robust stability to a multiplicative output perturbation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

Controller Design

Requirements 1 and 3 are valid and important at low frequencies, $0 \leq \omega \leq \omega_l \leq \omega_B$. Requirements 2, 4, 5 and 6 are conditions which are valid and important at high frequencies, $\omega_B \leq \omega_h \leq \omega \leq \infty$.

At frequencies where we want high gains (at low frequencies) the "worst-case" direction is related to $\underline{\sigma}(L)$, whereas at frequencies where we want low gains (at high frequencies) the "worst-case" direction is related to $\bar{\sigma}(L)$.