System Identification and Robust Control

Lecture 7: Introduction to Multivariable Control

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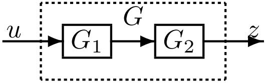
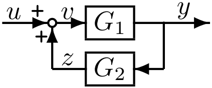
Transfer functions for MIMO systems [3.2]



(a) Cascade system



(b) Positive feedback system

Figure 1: Block diagrams for the cascade rule and the feedback rule

1. **Cascade rule - (Figure 1(a):**

$$G = G_2 G_1$$

2. **Feedback rule - (Figure 1(b)):**

$$v = (I - L)^{-1} u, \quad L = G_2 G_1$$

3. **Push-through rule:**

$$G_1 (I - G_2 G_1)^{-1} = (I - G_1 G_2)^{-1} G_1$$

MIMO Rule: *Start from the output, move backwards. If you exit from a feedback loop then include a term* $(I - L)^{-1}$ *where* L *is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).*

Example:

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \tag{1}$$

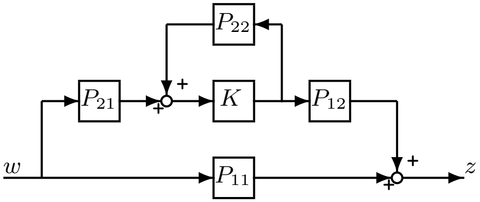


Figure 2: Block diagram corresponding to (1)

Negative feedback control systems

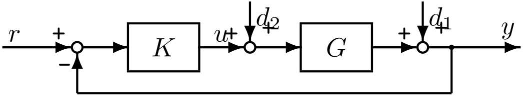


Figure 3: Conventional negative feedback control system

- L is the loop transfer function when breaking the loop at the plant *output*:

$$L = GK \tag{2}$$

Accordingly

$$S \triangleq (I+L)^{-1} \quad \textit{output sensitivity} \tag{3}$$

$$T \triangleq I - S = (I+L)^{-1}L = L(I+L)^{-1} \quad \textit{output complementary sensitivity} \tag{4}$$

$L_O \equiv L$, $S_O \equiv S$ and $T_O \equiv T$.

- L_I is the loop transfer function at the *input* to the plant:

$$L_I = KG \tag{5}$$

Input sensitivity:

$$S_I \triangleq (I + L_I)^{-1}$$

Input complementary sensitivity:

$$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$

- Some relationships:

$$(I + L)^{-1} + (I + L)^{-1}L = S + T = I \tag{6}$$

$$G(I + KG)^{-1} = (I + GK)^{-1}G \tag{7}$$

$$GK(I + GK)^{-1} = G(I + KG)^{-1}K = (I + GK)^{-1}GK \tag{8}$$

$$T = L(I + L)^{-1} = (I + L^{-1})^{-1} = (I + L)^{-1}L \tag{9}$$

Rule to remember: "G comes first and then G and K alternate in sequence".

Multivariable frequency response [3.3]

$$
\begin{aligned}
G(s) &= \text{transfer (function) matrix} \\
G(j\omega) &= \text{complex matrix representing response} \\
&\quad\ \text{to sinusoidal signal of frequency } \omega
\end{aligned}
$$

Note: $d \in R^m$ and $y \in R^l$

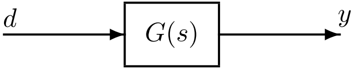


Figure 4: System $G(s)$ with input d and output y

$$y(s) = G(s)d(s) \tag{10}$$

Sinusoidal input to channel j

$$d_j(t) = d_{j0} \sin(\omega t + \alpha_j) \tag{11}$$

starting at $t = -\infty$. Output in channel i is a sinusoid with the same frequency

$$y_i(t) = y_{i0} \sin(\omega t + \beta_i) \tag{12}$$

Amplification (gain):

$$\frac{y_{io}}{d_{jo}} = |g_{ij}(j\omega)| \tag{13}$$

Phase shift:

$$\beta_i - \alpha_j = \angle g_{ij}(j\omega) \tag{14}$$

$g_{ij}(j\omega)$ represents the sinusoidal response from input j to output i.

Example: 2×2 multivariable system, sinusoidal signals of the same frequency ω to the two input channels:

$$d(t) = \left[\begin{array}{c} d_1(t) \\ d_2(t) \end{array} \right] = \left[\begin{array}{c} d_{10} \sin(\omega t + \alpha_1) \\ d_{20} \sin(\omega t + \alpha_2) \end{array} \right] \tag{15}$$

The output signal

$$y(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right] = \left[\begin{array}{c} y_{10} \sin(\omega t + \beta_1) \\ y_{20} \sin(\omega t + \beta_2) \end{array} \right] \tag{16}$$

can be computed by multiplying complex matrix $G(j\omega)$ by complex vector $d(\omega)$:

$$\begin{aligned} y(\omega) &= G(j\omega) d(\omega) \\ y(\omega) &= \left[\begin{array}{c} y_{10} e^{j\beta_1} \\ y_{20} e^{j\beta_2} \end{array} \right], \; d(\omega) = \left[\begin{array}{c} d_{10} e^{j\alpha_1} \\ d_{20} e^{j\alpha_2} \end{array} \right] \end{aligned} \tag{17}$$

Directions in multivariable systems [3.3.2]

SISO system ($y = Gd$, d and y are scalars):
The gain at a given frequency ω is

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on ω, but is independent of $|d(\omega)|$.

MIMO system ($y = Gd$, d and y are vectors):
We need to "sum up" magnitudes of elements in each vector by use of some norm

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \tag{18}$$

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots} \tag{19}$$

The *gain* of the system $G(s)$ is

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}} \tag{20}$$

The gain depends on ω, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the *direction* of the input d.

Example: Consider the five inputs (all $\|d\|_2 = 1$)

$$d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, d_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix},\ d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

For the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{21}$$

The five inputs d_j lead to the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},\ y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix},\ y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

with the 2-norms (i.e. the gains for the five inputs)

$$\|y_1\|_2 = 5.83,\ \|y_2\|_2 = 4.47,\ \|y_3\|_2 = 7.30, \|y_4\|_2 = 1.00,\ \|y_5\|_2 = 0.28$$

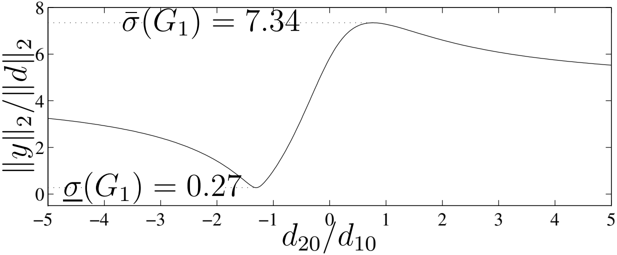


Figure 5: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (21)

The maximum value of the gain in (20) as the direction of the input is varied, is the maximum singular value of G,

$$\max_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \tag{22}$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \tag{23}$$

Eigenvalues are a poor measure of gain [3.3.3]

Example:

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \;\; G\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{24}$$

Both eigenvalues are equal to zero, but gain is equal to 100.

Problem: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).

Let d be an eigenvector of G, then $y = Gd = \lambda d$. Then,

$$\|y\|/\|d\| = \|\lambda d\|/\|d\| = |\lambda|$$

For generalizations of $|G|$ when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties must be satisfied:

Triangle inequality:

$$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\| \tag{25}$$

Multiplicative property:

$$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\| \tag{26}$$

As we may expect, the magnitude of the largest eigenvalue, $\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm.
Several matrix norms are possible (see Book's Appendix)

- Frobenius norm $\|G\|_F$
- Sum norm $\|G\|_{sum}$
- Maximum column sum $\|G\|_{i1}$
- Maximum row sum $\|G\|_{i\infty}$
- Maximum singular value $\|G\|_{i2} = \bar{\sigma}(G)$

The latter three norms are induced by a vector norm (reason for subscript i). The induced 2-norm (maximum singular value) is of special interest because it is the the ratio of the energy of the input/output signals.

Singular value decomposition [3.3.4]

Any matrix G may be decomposed into its singular value decomposition,

$$G = U\Sigma V^H \tag{27}$$

- Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal;
- U is an $l \times l$ unitary matrix of output singular vectors, u_i,
- V is an $m \times m$ unitary matrix of input singular vectors, v_i,

$$\sigma_i(G) = \sqrt{\lambda_i(G^H G)} = \sqrt{\lambda_i(G G^H)} \tag{28}$$

$$(GG^H)U = U\Sigma\Sigma^H, \qquad (G^H G)V = V\Sigma^H\Sigma \tag{29}$$

Input and output directions. The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

$$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1 \tag{30}$$

$$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j \tag{31}$$

The column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*.

$$G = U\Sigma V^H \Rightarrow GV = U\Sigma \quad (V^H V = I) \Rightarrow Gv_i = \sigma_i u_i \tag{32}$$

If we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ σ_i gives the gain of the matrix G in this direction.

$$\sigma_i(G) = \|Gv_i\|_2 = \frac{\|Gv_i\|_2}{\|v_i\|_2} \tag{33}$$

Maximum and minimum singular values. The largest gain for *any* input direction is

$$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} \tag{34}$$

The smallest gain for any input direction is

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} \tag{35}$$

where $k = \min\{l, m\}$. For any vector d we have

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \tag{36}$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \tag{37}$$

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the "strongest", "high-gain" or "most important" directions.

Example:

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{38}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$, the smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. Since in (38) both inputs affect both outputs, we say that the system is *interactive*.

The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*; $\bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example: **Distillation process.** Steady-state model of a distillation column

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{39}$$

Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{40}$$

The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$. For dynamic systems the singular values and their associated directions vary with frequency (Figure **??**).

Singular values for performance [3.3.5]

SVD provides insight into the directionality of MIMO systems. But the maximum singular value is very useful in terms of frequency-domain performance and robustness. We here consider performance.

Performance measure for SISO systems:

$$|e(\omega)|/|r(\omega)| = |S(j\omega)|$$

.

Generalization for MIMO systems $\|e(\omega)\|_2/\|r(\omega)\|_2$

$$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega)) \tag{41}$$

Introduction to Multivariable Control

For *performance* we want gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ small for any direction of $r(\omega)$

$$\begin{aligned}\bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|,\ \forall\omega \quad &\Leftrightarrow \quad \bar{\sigma}(w_P S) < 1, \forall\omega \\ &\Leftrightarrow \quad \|w_P S\|_\infty < 1 \end{aligned} \tag{42}$$

where the $\mathcal{H}_\infty$ norm is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega)) \tag{43}$$

Typical singular values of $S(j\omega)$ in Figure 6.

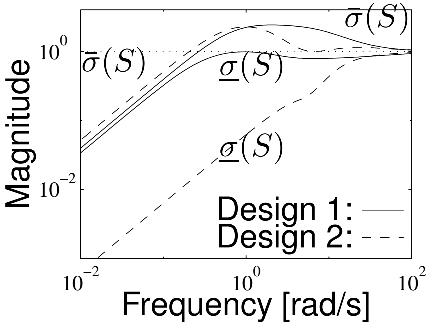


Figure 6: Singular values of S for a 2×2 plant with RHP-zero

- *Bandwidth*, ω_B: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

  It is then understood that the bandwidth is at least ω_B for any direction of the input (reference or disturbance) signal.

  Since $S = (I + L)^{-1}$, the singular values inequality $\underline{\sigma}(A) - 1 \leq \frac{1}{\bar{\sigma}(I+A)^{-1}} \leq \underline{\sigma}(A) + 1$ yields

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \tag{44}$$

- low ω : $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$
- high ω: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$

Poles [4.4]

Definition: The poles p_i of a system with state-space description

$$\dot{x} = Ax + Bu \quad (45)$$

$$y = Cx + Du \quad (46)$$

are the eigenvalues $\lambda_i(A), i = 1, \ldots, n$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as $\phi(s) \stackrel{\Delta}{=} \det(sI - A) = \prod_{i=1}^{n}(s - p_i)$. Thus the poles are the roots of the characteristic equation

$$\phi(s) \stackrel{\Delta}{=} \det(sI - A) = 0 \quad (47)$$

Poles and stability [4.4.1]

Theorem

A linear dynamic system $\dot{x} = Ax + Bu$ is stable if and only if all the poles are in the open left-half plane (LHP), that is, $\Re\{\lambda_i(A)\} < 0, \forall i$. A matrix A with such a property is said to be "stable" or Hurwitz.

Poles from transfer functions [4.4.3]

Theorem

The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$, is the least common denominator of all non-identically-zero minors of all orders of $G(s)$.

- A minor of a matrix is the determinant of the matrix obtained by deleting certain rows and/or columns of the matrix.
- The notation M_c^r is used to denote the minor corresponding to the deletion of rows r and columns c in $G(s)$.

Example [4.7]: Consider the 2×2 system, with 2 inputs and 2 outputs,

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \tag{48}$$

The minors of order 1 are the four elements all have $(s+1)(s+2)$ in the denominator.

Minor of order 2

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \tag{49}$$

Least common denominator of all the minors:

$$\phi(s) = (s+1)(s+2) \tag{50}$$

Minimal realization has two poles: $s = -1$; $s = -2$.

Example [4.8]: Consider the 2×3 system, with 3 inputs and 2 outputs,

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} *$$

$$* \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix} \tag{51}$$

Minors of order 1:

$$\frac{1}{s+1}, \; \frac{s-1}{(s+1)(s+2)}, \; \frac{-1}{s-1}, \; \frac{1}{s+2}, \; \frac{1}{s+2} \tag{52}$$

Minor of order 2 corresponding to the deletion of column 2:

$$M_2 = \frac{(s-1)(s+2)(s-1)(s+1) + (s+1)(s+2)(s-1)^2}{((s+1)(s+2)(s-1))^2} =$$

$$= \frac{2}{(s+1)(s+2)} \tag{53}$$

The other two minors of order two are

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}, \quad M_3 = \frac{1}{(s+1)(s+2)} \tag{54}$$

Least common denominator:

$$\phi(s) = (s+1)(s+2)^2(s-1) \tag{55}$$

The system therefore has four poles: $s = -1$, $s = 1$ and two at $s = -2$.

Note MIMO-poles are essentially the poles of the elements. A procedure is needed to determine multiplicity.

Zeros [4.5]

SISO system: Zeros z_i are the solutions to $G(z_i) = 0$.

Example:

$$Y = \frac{s+2}{s^2+7s+12}U$$

Compute the response when

$$\begin{aligned}
u(t) &= e^{-2t},\ y(0) = 0,\ \dot{y}(0) = -1 \\
\mathcal{L}\{u(t)\} &= \frac{1}{s+2} \\
s^2Y &- sy(0) - \dot{y}(0) + 7sY - 7y(0) + 12Y = 1 \\
s^2Y &+ 7sY + 12Y + 1 = 1 \\
&\Rightarrow Y(s) = 0
\end{aligned}$$

Assumption: $g(s)$ has a zero z, $g(z) = 0$. Then for input $u(t) = u_0 e^{zt}$ the output is $y(t) \equiv 0$, $t > 0$. (with appropriate initial conditions)

In general, zeros are values of s at which $G(s)$ loses rank.

Zeros from state-space realizations [4.5.1]

The state-space equations of a system can be written as:

$$P(s)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}, \qquad P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \tag{56}$$

The zeros are then the values $s = z$ for which the polynomial system matrix $P(s)$ loses rank, resulting in zero output for some non-zero input

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_z \\ u_z \end{bmatrix} = 0$$

Non-trivial solution: The zeros are the solutions of

$$\det\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = 0$$

MATLAB:

```
zero = tzero(A,B,C,D)
```

Zeros from transfer functions [4.5.2]

Definition: z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{n_z}(s - z_i)$ where n_z is the number of finite zeros of $G(s)$.

Theorem

The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominators.

Remark. This gives the same result as the Smith McMillan form introduced by Rosenbrock (See Section 4.6).

Example [4.9]:

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix} \tag{57}$$

The normal rank of $G(s)$ is 2.
Minor of order 2: $\det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = 2\frac{s-4}{s+2}$.
Pole polynomial: $\phi(s) = s+2$.
Zero polynomial: $z(s) = s-4$.

Note: Multivariable zeros have in general no relationship with the zeros of the transfer function elements.

Example [4.7 Continued]:

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \tag{58}$$

The normal rank of $G(s)$ is 2.
Minor of order 2: $\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)}$
Pole polynomial: $\phi(s) = 1.25^2(s+1)(s+2)$
Zero polynomial: $z(s) = 1 \Rightarrow$ no multivariable zeros.

Example [4.10]:

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix} \tag{59}$$

The normal rank of $G(s)$ is 1.
No value of s for which both elements of $G(s)$ (minors of order 1) become zero $\Rightarrow G(s)$ has no zeros.

Directions of poles and zeros

Let $G(s) = C(sI - A)^{-1}B + D$.

Zero directions [4.5.3]. Let $G(s)$ have a zero at $s = z$. Then $G(s)$ loses rank at $s = z$, and there exist non-zero vectors u_z and y_z such that

$$G(z)u_z = 0, \quad y_z^H G(z) = 0 \tag{60}$$

$u_z =$ input zero direction ($u_z^H u_z = 1$)
$y_z =$ output zero direction ($y_z^H y_z = 1$)
y_z gives information about which output (or combination of outputs) may be difficult to control.

Pole directions [4.4.4]. Let $G(s)$ have a pole at $s = p$. Then $G(p)$ is infinite, and we may write

$$G(p)u_p = \infty, \quad y_p^H G(p) = \infty \tag{61}$$

$u_p =$ input pole direction $(u_p^H u_p = 1)$
$y_p =$ output pole direction $(y_p^H y_p = 1)$

Singular Value Decomposition. The zero/pole directions may in principle be obtained from an SVD of

$$G(z/p) = U\Sigma V^H$$

$u_z =$ last column in V, $y_z =$ last column of U
(corresponding to the zero singular value of $G(z)$)
$u_p =$ first column in V, $y_p =$ first column of U
(corresponding to the infinite singular value of $G(p)$)

Example: Plant in (57) has a RHP-zero at $z = 4$ and a LHP-pole at $p = -2$.

$$\begin{aligned} G(z) &= G(4) = \frac{1}{6}\begin{bmatrix} 3 & 4 \\ 4.5 & 6 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 0.55 & -0.83 \\ 0.83 & 0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}^H \end{aligned} \tag{62}$$

The zero input and output directions are associated with the zero singular value of $G(z)$, i.e.

$$u_z = \begin{bmatrix} -0.80 \\ 0.60 \end{bmatrix} \quad y_z = \begin{bmatrix} -0.83 \\ 0.55 \end{bmatrix} \tag{63}$$

For pole directions consider

$$G(p+\epsilon) = G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -3+\epsilon & 4 \\ 4.5 & 2(-3+\epsilon) \end{bmatrix} \tag{64}$$

The SVD as $\epsilon \to 0$ yields

$$G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -0.55 & -0.83 \\ 0.83 & -0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}^H$$

$$u_p = \begin{bmatrix} 0.60 \\ -0.80 \end{bmatrix} \quad y_p = \begin{bmatrix} -0.55 \\ 0.83 \end{bmatrix} \tag{65}$$

Note 1: Locations of poles and zeros are independent of input and output scalings, their directions are *not*. Thus, the inputs and outputs need to be scaled properly before making any interpretations based on pole and zero directions.

Note 2: This method is not numerically reliable.

Remarks on poles and zeros [4.6]

1. For square systems the poles and zeros of $G(s)$ are "essentially" the poles and zeros of $\det G(s)$.

   This fails when zero and pole in different parts of the system cancel when forming $\det G(s)$.

$$G(s) = \begin{bmatrix} (s+2)/(s+1) & 0 \\ 0 & (s+1)/(s+2) \end{bmatrix} \tag{66}$$

   $\det G(s) = 1$, although the system obviously has poles at -1 and -2 and (multivariable) zeros at -1 and -2.

2. System (66) has poles and zeros at the same locations (at -1 and -2). Their directions are different. They do not cancel or otherwise interact.

3. There are no zeros if the outputs contain direct information about all the states; that is, if from y we can directly obtain x (e.g. $C = I$ and $D = 0$).

4. Zeros usually appear when there are fewer inputs or outputs than states

5. **Moving poles.** (a) Feedback control ($G(I + KG)^{-1}$) moves the poles. (b) Series compensation (GK, feedforward control) can cancel poles in G by placing zeros in K (but not move them). (c) Parallel compensation ($G + K$) cannot affect the poles in G.

6. **Moving zeros.** (a) With feedback, the zeros of $G(I + KG)^{-1}$ are the zeros of G plus the poles of K. , i.e. the zeros are unaffected by feedback. (b) Series compensation can counter the effect of zeros in G by placing poles in K to cancel them, but cancellations are not possible for RHP-zeros due to internal stability (see Section 40). (c) The only way to move zeros is by parallel compensation, $y = (G + K)u$, which, if y is a physical output, can only be accomplished by adding an extra input (actuator).

Stability [4.3]

Definition

A system is **(internally) stable** if none of its components contains hidden unstable modes and the injection of bounded external signals at any place in the system results in bounded output signals measured anywhere in the system. The word "internal" implies that <u>all</u> the states must be stable not only inputs/outputs.

Definition

State stabilizable, state detectable and hidden unstable modes. A system is state stabilizable if all unstable modes are state controllable. A system is state detectable if all unstable modes are state observable. A system with unstabilizable or undetectable modes is said to contain hidden unstable modes.

Internal stability of feedback systems [4.7]

Note: Checking the pole of S or T is not sufficient to determine internal stability

Example [4.14]: (Figure 7). In forming $L = GK$ we cancel the term $(s-1)$ (a RHP pole-zero cancellation) to obtain

$$L = GK = \frac{k}{s}, \text{ and } S = (I+L)^{-1} = \frac{s}{s+k} \tag{67}$$

$S(s)$ is stable, i.e. transfer function from d_y to y is stable. However, the transfer function from d_y to u is unstable:

$$u = -K(I+GK)^{-1} d_y = -\frac{k(s+1)}{(s-1)(s+k)} d_y \tag{68}$$

Consequently, although the system appears to be stable when considering the output signal y, it is unstable when considering the "internal" signal u, so the system is (internally) unstable.

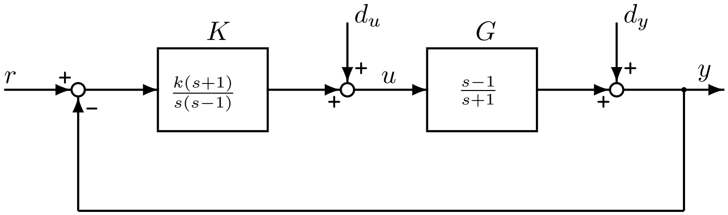


Figure 7: Internally unstable system

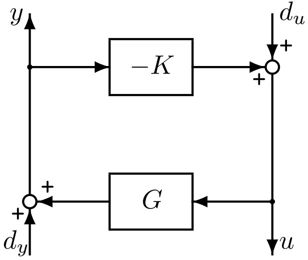


Figure 8: Block diagram used to check internal stability of feedback system

For *internal* stability consider

$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y \tag{69}$$

$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y \tag{70}$$

Theorem

The feedback system in Figure 8 is **internally stable** *if and only if all four closed-loop transfer matrices in (69) and (70) are stable.*

Theorem

Assume there are no RHP pole-zero cancellations between $G(s)$ and $K(s)$, i.e. all RHP-poles in $G(s)$ and $K(s)$ are contained in the minimal realizations of GK and KG. Then, the feedback system in Figure 8 is internally stable if and only if <u>one</u> of the four closed-loop transfer function matrices in (69) and (70) is stable.

Implications of the internal stability requirement

1. *If $G(s)$ has a RHP-zero at z, then $L = GK$, $T = GK(I+GK)^{-1}$, $SG = (I+GK)^{-1}G$, $L_I = KG$ and $T_I = KG(I+KG)^{-1}$ will each have a RHP-zero at z.*

2. *If $G(s)$ has a RHP-pole at p, then $L = GK$ and $L_I = KG$ also have a RHP-pole at p, while $S = (I+GK)^{-1}, KS = K(I+GK)^{-1}$ and $S_I = (I+KG)^{-1}$ have a RHP-zero at p.*

Introduction to MIMO robustness [3.7]

Motivating robustness example 1: Spinning Satellite [3.7.1]

Angular velocity control of a satellite spinning about one of its principal axes:

$$G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}; \quad a = 10 \tag{71}$$

A minimal, state-space realization, $G = C(sI - A)^{-1}B + D$, is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right] \tag{72}$$

Poles at $s = \pm ja$ For stabilization:

$$K = I$$

$$T(s) = GK(I + GK)^{-1} = \frac{1}{s+1} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \tag{73}$$

Nominal stability (NS). Two closed loop poles at $s = -1$ and

$$A_{cl} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nominal performance (NP). Figure 9(a)

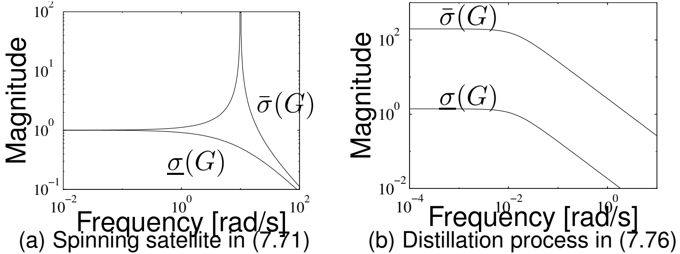


Figure 9: Typical plots of singular values

- $\underline{\sigma}(L) \leq 1 \quad \forall\omega$ poor performance in low gain direction
- g_{12}, g_{21} large $\Rightarrow$ strong interaction

Robust stability (RS).

Check stability: one loop at a time.

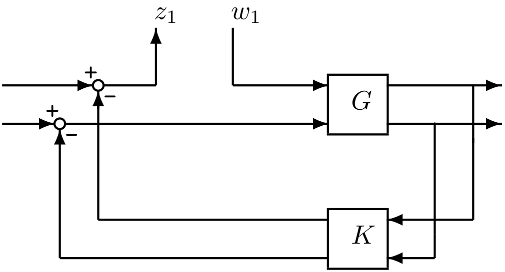


Figure 10: Checking stability margins "one-loop-at-a-time"

$$\frac{z_1}{w_1} \triangleq L_1(s) = \frac{1}{s} \Rightarrow GM = \infty, PM = 90^\circ \tag{74}$$

Note that $T_{11}(s) = \frac{1}{s+1} = \frac{L_1(s)}{1+L_1(s)} \Leftrightarrow L_1(s) = \frac{1}{s}$

- Good Robustness? This result suggests YES ... but NO
- Consider perturbation in each feedback channel

$$u_1' = (1+\epsilon_1)u_1, \quad u_2' = (1+\epsilon_2)u_2 \tag{75}$$

$$B' = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$$

Closed-loop state matrix:

$$A_{cl}' = A - B'KC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

Characteristic polynomial:

$$\det(sI - A'_{cl}) = s^2 + \underbrace{(2 + \epsilon_1 + \epsilon_2)}_{a_1} s + \\ + \underbrace{1 + \epsilon_1 + \epsilon_2 + (a^2 + 1)\epsilon_1\epsilon_2}_{a_0}$$

The perturbed system is stable if and only if both the coefficients a_1 and a_0 are positive. The system is always stable if we consider uncertainty in only one channel at a time: $(-1 < \epsilon_1 < \infty, \epsilon_2 = 0)$ and $(\epsilon_1 = 0, -1 < \epsilon_2 < \infty)$. This confirms GM$=\infty$.

But only *small simultaneous changes* in the two channels: for example, let $\epsilon_1 = -\epsilon_2$, then the system is unstable $(a_0 < 0)$ for

$$|\epsilon_1| > \frac{1}{\sqrt{a^2 + 1}} \approx 0.1$$

Summary. Checking single-loop margins is inadequate for MIMO problems.

Motivating robustness example 2: Distillation Process [3.7.2]

Idealized dynamic model of a distillation column,

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{76}$$

(time is in minutes).

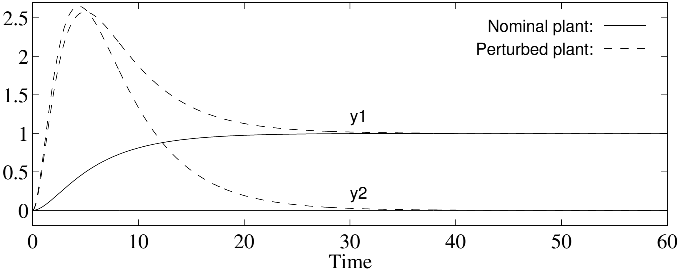


Figure 11: Response with decoupling controller to filtered reference input $r_1 = 1/(5s+1)$. The perturbed plant has 20% gain uncertainty as given by (79).

Inverse-based controller or equivalently steady-state decoupler with a PI controller ($k_1 = 0.7$)

$$K_{\mathrm{inv}}(s) = \frac{k_1}{s} G^{-1}(s) = \frac{k_1(1+75s)}{s} \begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \tag{77}$$

Nominal performance (NP).

$$GK_{\mathrm{inv}} = K_{\mathrm{inv}}G = \frac{0.7}{s} I$$

First order response with time constant $1/0.7 = 1.43$ (Fig. 11).
Nominal performance (NP) achieved with decoupling controller.

Robust stability (RS).

$$S = S_I = \frac{s}{s+0.7} I; \quad T = T_I = \frac{1}{1.43s+1} I \tag{78}$$

In each channel: GM$=\infty$, PM$=90^\circ$.

Input gain uncertainty (75) with $\epsilon_1 = 0.2$ and $\epsilon_2 = -0.2$:

$$u_1' = 1.2u_1, \quad u_2' = 0.8u_2 \tag{79}$$

$$\begin{aligned} L_I'(s) &= K_{\text{inv}} G' = K_{\text{inv}} G \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} = \\ & \frac{0.7}{s} \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \end{aligned} \tag{80}$$

Perturbed closed-loop poles are

$$s_1 = -0.7(1+\epsilon_1), \quad s_2 = -0.7(1+\epsilon_2) \tag{81}$$

Closed-loop stability as long as the input gains $1+\epsilon_1$ and $1+\epsilon_2$ remain positive
$\Rightarrow$ *Robust stability (RS) achieved with respect to input gain errors for the decoupling controller.*

Robust performance (RP).
Performance with model error poor (Fig. 11)

- SISO: NP+RS $\Rightarrow$ RP
- MIMO: NP+RS $\not\Rightarrow$ RP

RP is not achieved by the decoupling controller.

Robustness conclusions [3.7.3]

Multivariable plants can display a sensitivity to uncertainty (in this case input uncertainty) which is fundamentally different from what is possible in SISO systems.