System Identification and Robust Control

Lecture 6: Uncertainty and Robustness for SISO Systems

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Introduction [7.1]

A control system is robust if it is insensitive to differences between the actual system and the system model used to design the controller. These differences are referred to as *model/plant mismatch* or simply *model uncertainty*.

Our approach is:

1. Determine the uncertainty set: find a mathematical representation of the model uncertainty ("clarify what we know about what we don't know").
2. Check Robust stability (RS): determine whether the system remains stable for all plants in the uncertainty set.
3. Check Robust performance (RP): if RS is satisfied, determine whether the performance specifications are met for all plants in the uncertainty set.

Notation:

Π – a set of possible perturbed plant models ("uncertainty set").

$G(s) \in \Pi$ – nominal plant model (with no uncertainty).

$G_p(s) \in \Pi$ and $G'(s) \in \Pi$ – particular perturbed plant models.

Classes of uncertainty [7.2]

1. **Parametric uncertainty.** Here the structure of the model (including the order) is known, but some of the parameters are uncertain. For instance, $\alpha_p \in [\alpha_{min}, \alpha_{max}]$. That is, we have parameter sets of the form

$$\alpha_p = \bar{\alpha}(1 + r_\alpha \Delta)$$

   where $\bar{\alpha} = (\alpha_{max} + \alpha_{min})/2$, $r_\alpha = (\alpha_{max} - \alpha_{min})/(\alpha_{max} + \alpha_{min})$ and Δ is any real scalar satisfying $|\Delta| \leq 1$.

2. **Neglected and unmodelled dynamics uncertainty.** Here the model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process. Any model of a real system will contain this source of uncertainty.

3. **Lumped uncertainty.** Here the uncertainty description represents one or several sources of parametric and/or unmodelled dynamics uncertainty combined into a single lumped perturbation of a chosen structure.

- Parametric uncertainty is sometimes called structured uncertainty as it models the uncertainty in a structured manner.
- Analogously, lumped dynamics uncertainty is sometimes called unstructured uncertainty.
- The frequency domain is well suited for describing both *neglected/unmodelled dynamics* and *lumped* uncertainties.

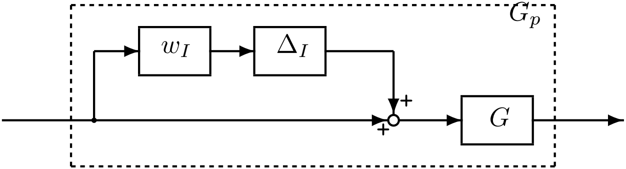

Figure 1: Plant with multiplicative uncertainty

Uncertainty in SISO Systems

Multiplicative uncertainty:

$$\Pi_I : \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s));$$

where

$$\underbrace{|\Delta_I(j\omega)| \leq 1 \ \forall\omega}_{\|\Delta_I\|_\infty \leq 1} \tag{1}$$

Here $\Delta_I(s)$ is *any* stable transfer function which at each frequency is less than or equal to one in magnitude. Some allowable $\Delta_I(s)$'s

$$\frac{s-z}{s+z}, \quad \frac{1}{\tau s + 1}, \quad \frac{1}{(5s+1)^3}, \quad \frac{0.1}{s^2 + 0.1s + 1}$$

Inverse multiplicative uncertainty

$$\Pi_{iI} : \quad G_p(s) = G(s)(1 + w_{iI}(s)\Delta_{iI}(s))^{-1}; \quad |\Delta_{iI}(j\omega)| \leq 1 \ \forall\omega$$

Representing uncertainty in the frequency domain [7.4]
Uncertainty regions [7.4.1]

$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \le k, \theta, \tau \le 3 \tag{2}$$

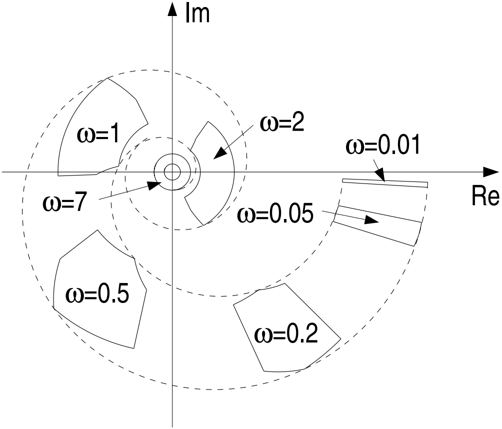

Figure 2: Uncertainty regions of the Nyquist plot at given frequencies. Data from (2)

Uncertainty in SISO Systems

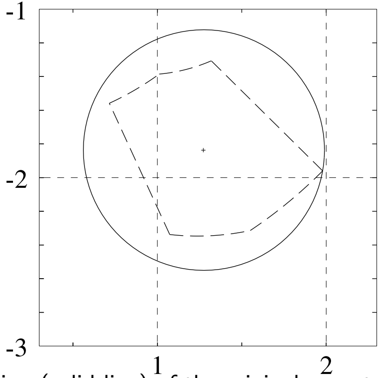

Figure 3: Disc approximation (solid line) of the original uncertainty region (dashed line). Plot corresponds to $\omega = 0.2$ in Figure 2

Approximation by complex perturbations [7.4.2]

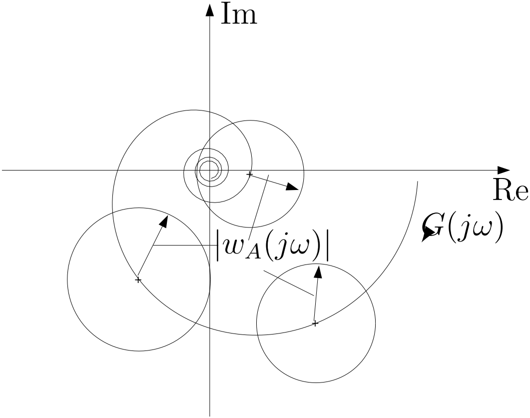

Figure 4: Disc-shaped uncertainty regions generated by complex additive uncertainty, $G_p = G + w_A\Delta$

Uncertainty in SISO Systems

We use disc-shaped regions to represent uncertainty regions (Figures 3 and 4) generated by

$$\Pi_A : \quad G_p(s) = G(s) + w_A(s)\Delta_A(s); \quad |\Delta_A(j\omega)| \leq 1 \ \forall\omega \tag{3}$$

where $\Delta_A(s)$ is *any* stable transfer function which at each frequency is no larger than one in magnitude.

Alternative: *multiplicative uncertainty* description as in (1),

$$\Pi_I : \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s)); \quad |\Delta_I(j\omega)| \leq 1, \forall\omega \tag{4}$$

(3) and (4) are equivalent if at each frequency

$$|w_I(j\omega)| = |w_A(j\omega)|/|G(j\omega)| \tag{5}$$

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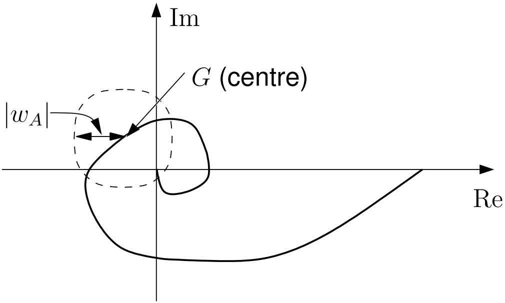

Figure 5: The set of possible plants includes the origin at frequencies where $|w_A(j\omega)| \geq |G(j\omega)|$, or equivalently $|w_I(j\omega)| \geq 1$

- At these frequencies we do not know the phase of the plant, and we allow for zeros crossing from the left to the right-half plane.
- To see this, consider a frequency where $|w_I(j\omega_o)| \geq 1$. Then there exists a $|\Delta_I| \leq 1$ such that $G_p(j\omega_o) = 0$ in (4), that is, there exists a possible plant with zeros at $s = \pm j\omega_o$. For this plant at frequency ω_o the input has no effect on the output, so control has no effect (tight control is not possible).

Obtaining the weight for complex uncertainty [7.4.3]

1. Select a nominal model $G(s)$.

2. *Additive uncertainty.* At each frequency find the smallest radius $l_A(\omega)$ which includes all the possible plants Π:

$$|w_A(jw)| \geq l_A(\omega) = \max_{G_P \in \Pi} |G_p(j\omega) - G(j\omega)| \tag{6}$$

3. *Multiplicative (relative) uncertainty* (preferred uncertainty form).

$$|w_I(jw)| \geq l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right| \tag{7}$$

Example

Multiplicative weight for parametric uncertainty (Example 7.3) *(lecture06a.m)*
Consider again the set of plants with parametric uncertainty given in (2)

$$\Pi : \quad G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3 \tag{8}$$

We want to represent this set using multiplicative uncertainty with a rational weight $w_I(s)$*. We select a delay-free nominal model*

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5s + 1} \tag{9}$$

We plot $|(G_p - G)/G|$ *as function of the frequency.*

Uncertainty in SISO Systems

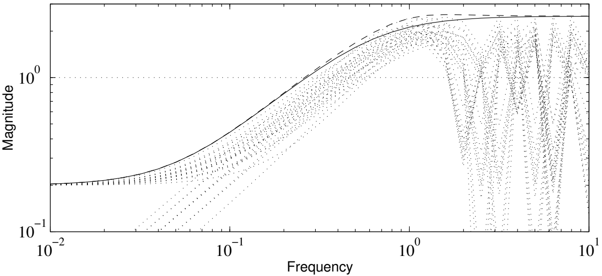

Figure 6: Relative errors for 27 combinations of k, τ and θ with delay-free nominal plant (dotted lines). Solid line: First-order weight $|w_{I1}|$ in (10). Dashed line: Third-order weight $|w_I|$ in (11)

$$w_{I1}(s) = \frac{Ts + 0.2}{(T/2.5)s + 1}, \quad T = 4 \tag{10}$$

$$w_I(s) = \omega_{I1}(s)\ \frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1} \tag{11}$$

SISO Robust stability [7.5]

- We have so far discussed how to represent the uncertainty mathematically.
- We derive now conditions that will ensure that the system remains stable and satisfies performance requirements for all perturbations in the uncertainty set.

RS with multiplicative uncertainty

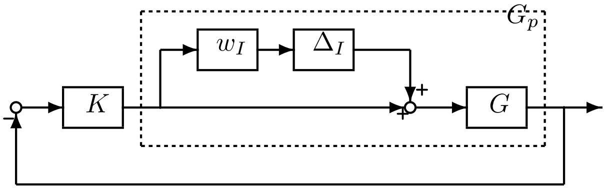

Figure 7: Feedback system with multiplicative uncertainty

Graphical derivation of RS-condition.
In Figure 8 $|-1-L| = |1+L|$ is the distance from the point -1 to the centre of the disc representing L_p, $|w_I L|$ is the radius of the disc. Encirclements are avoided if none of the discs cover -1, and we get from Figure 8

$$\mathrm{RS} \quad = \quad |w_I L| < |1+L|, \quad \forall\omega \tag{12}$$

$$= \quad \left|\frac{w_I L}{1+L}\right| < 1, \forall\omega \Leftrightarrow |w_I T| < 1, \forall\omega \tag{13}$$

$$\overset{\text{def}}{\Leftrightarrow} \quad \|w_I T\|_\infty < 1 \tag{14}$$

$$\boxed{\mathrm{RS} : |T| < 1/|w_I|, \quad \forall\omega \iff \|w_I T\|_\infty < 1} \tag{15}$$

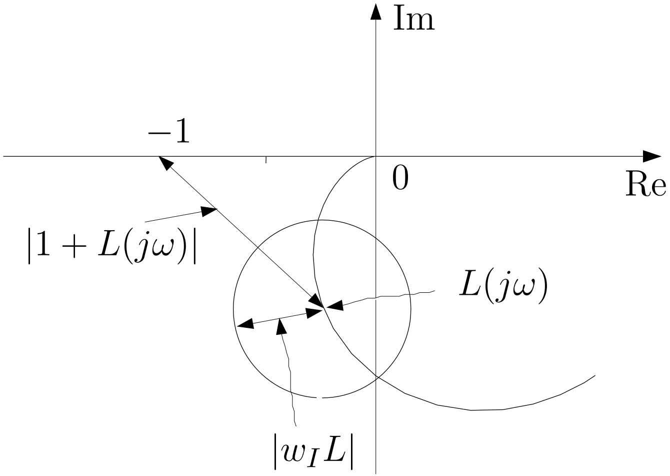

Figure 8: Nyquist plot of L_p for robust stability

Uncertainty in SISO Systems

Example

Robust stability (Example 7.6) *(lecture06b.m)*
Consider the following nominal plant and PI-controller

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)} \quad K(s) = K_c \frac{12.7s+1}{12.7s}$$

$K_c = K_{c1} = 1.13$ *(Ziegler-Nichols: See Lecture 5!). One "extreme" uncertain plant is* $G'(s) = 4(-3s+1)/(4s+1)^2$. *For this plant the relative error* $|(G' - G)/G|$ *is* 0.33 *at low frequencies; it is* 1 *at about* 0.1 *rad/s, and it is* 5.25 *at high frequencies* $\Rightarrow$ *uncertainty weight*

$$w_I(s) = \frac{10s + 0.33}{(10/5.25)s + 1}$$

which closely matches this relative error. We now want to evaluate whether the system remains stable for all possible plants. This is not the case as seen from Figure 9 where we see that the magnitude of the nominal complementary sensitivity function exceeds the bound, so (15) *is not satisfied. To achieve robust stability we need to reduce the controller gain.*

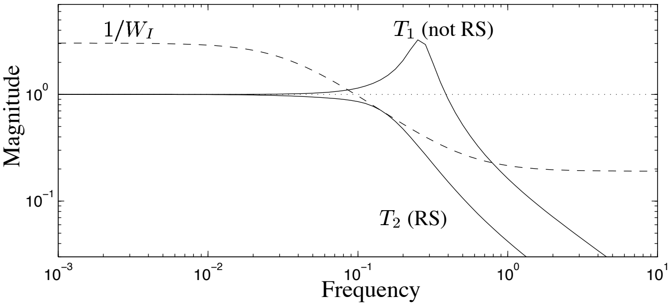

Figure 9: Checking robust stability with multiplicative uncertainty

By trial and error we find that reducing the gain to $K_c = K_{c2} = 0.31$ just achieves RS as seen from T_2 in Fig. 9.

Remark:

The procedure is *conservative*! For K_{c2}, system with the "extreme" plant is not at the limit of instability; we can increase gain to $k_{c2} = 0.58$ before we get instability.

Uncertainty in SISO Systems

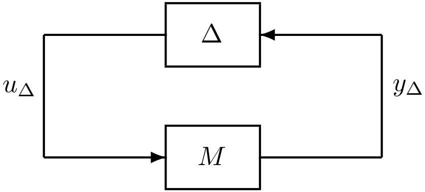

Figure 10: $M\Delta$-structure

$M\Delta$-structure derivation of RS-condition. The stability of the system in Figure 7 is equivalent to stability of the system in Figure 10, where $\Delta = \Delta_I$ and

$$M = w_I K(1 + GK)^{-1} G = w_I T \tag{16}$$

Uncertainty in SISO Systems

The Nyquist stability condition then determines RS if and only if the "loop transfer function" $M\Delta$ does not encircle -1 for all Δ. Thus,

$$\mathrm{RS} \quad = \quad |1 + M\Delta| > 0, \quad \forall\omega, \forall|\Delta| \leq 1 \tag{17}$$

$$\begin{aligned} \mathrm{RS} \quad &= \quad 1 - |M(j\omega)| > 0, \quad \forall\omega && (18) \\ &= \quad |M(j\omega)| < 1, \quad \forall\omega && (19) \end{aligned}$$

RS with inverse multiplicative uncertainty [7.5.3]

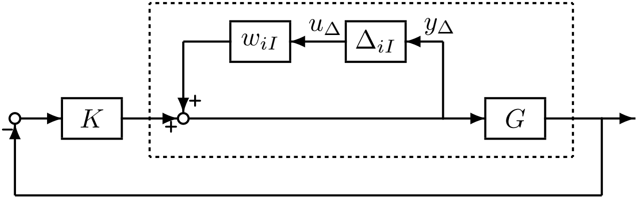

Figure 11: Feedback system with inverse multiplicative uncertainty

$$\boxed{\text{RS} \quad \Leftrightarrow \quad |S| < 1/|w_{iI}|, \quad \forall \omega} \tag{20}$$

SISO Robust performance [7.6]
Nominal performance in the Nyquist plot

$$\text{NP} \quad = \quad |w_P S| < 1 \quad \forall\omega \quad = \quad |w_P| < |1 + L| \quad \forall\omega \tag{21}$$

See Figure:

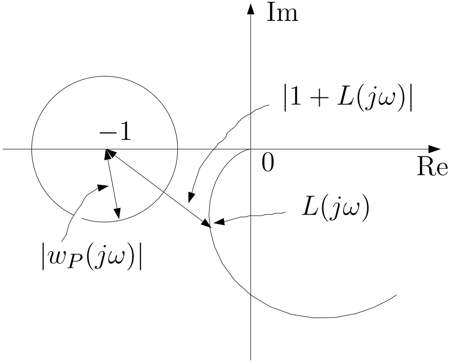

Figure 12: Nyquist plot illustration of nominal performance condition $|w_P| < |1 + L|$

Robust performance [7.6.2]

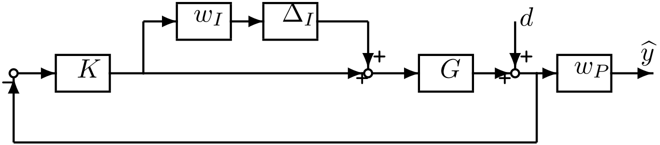

Figure 13: Diagram for robust performance with multiplicative uncertainty

Uncertainty in SISO Systems

For robust performance we require the performance condition (21) to be satisfied for *all* possible plants, that is, including the worst-case uncertainty.

$$\text{RP} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad |w_P S_p| < 1 \quad \forall S_p, \forall \omega \tag{22}$$

$$= \quad |w_P| < |1 + L_p| \quad \forall L_p, \forall \omega \tag{23}$$

This corresponds to requiring $|\widehat{y}/d| < 1 \ \forall \Delta_I$ in Figure 13, where we consider multiplicative uncertainty, and the set of possible loop transfer functions is

$$L_p = G_p K = L(1 + w_I \Delta_I) = L + w_I L \Delta_I \tag{24}$$

Graphical derivation of RP-condition. (Figure 14)

$$\begin{aligned} \text{RP} &= |w_P| + |w_I L| < |1+L|, \quad \forall\omega && (25)\\ &= |w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1, \forall\omega && (26) \end{aligned}$$

$$\boxed{\text{RP} \;=\; \max_\omega \left(|w_P S| + |w_I T|\right) < 1} \qquad (27)$$

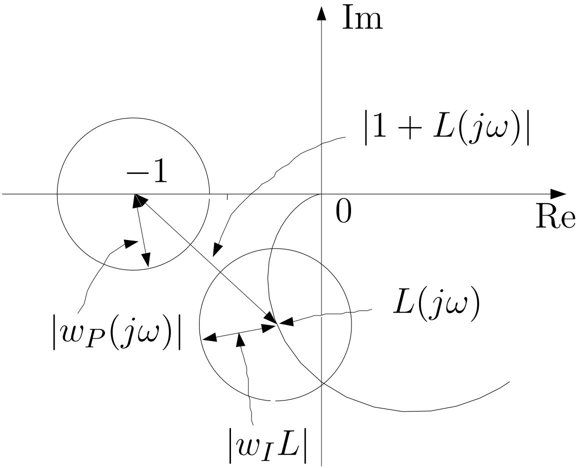

Figure 14: Nyquist plot illustration of robust performance condition $|w_P| < |1 + L_p|$

The relationship between NP, RS and RP [7.6.3]

$$\text{NP} = |w_P S| < 1, \forall\omega \quad (28)$$

$$\text{RS} = |w_I T| < 1, \forall\omega \quad (29)$$

$$\text{RP} = |w_P S| + |w_I T| < 1, \forall\omega \quad (30)$$

- A prerequisite for RP is that we satisfy NP and RS. This applies in general, both for SISO and MIMO systems and for any uncertainty.

Uncertainty in SISO Systems

- For SISO systems, if we satisfy both RS and NP, then we have at each frequency
$$|w_P S| + |w_I T| \leq 2 \max\{|w_P S|, |w_I T|\} < 2 \tag{31}$$
Therefore, within a factor of at most 2, we will automatically get RP when NP and RS are satisfied.
- Note that
$$|w_P S| + |w_I T| \geq \min\{|w_P|, |w_I|\} \tag{32}$$
We *cannot have both* $|w_P| > 1$ *(i.e. good performance) and* $|w_I| > 1$ *(i.e. more than* 100% *uncertainty) at the same frequency.*