System Identification and Robust Control

Lecture 5: Classical Feedback Control

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The control problem [1.2]:

$$y = Gu + G_d d \quad (1)$$

y	:	output/controlled variable
u	:	input/manipulated variable
d	:	disturbance
r	:	reference/setpoint

Regulator problem	:	counteract d
Servo problem	:	let y follow r

Goal of control: make control error $e = y - r$ "small".

Note: To arrive at a good design for the controller K we need *a priori* information about expected disturbances/references and knowledge of plant model G and disturbance model G_d.

Scaling [1.4]:

Proper scaling simplifies controller design and performance analysis.
Unscaled:

$$\widehat{y} = \widehat{G}\widehat{u} + \widehat{G}_d\widehat{d}; \quad \widehat{e} = \widehat{y} - \widehat{r} \tag{2}$$

SISO :
Scale $\widehat{d}$, $\widehat{u}$ by

- $\widehat{d}_{\max}$ — largest expected change in disturbance
- $\widehat{u}_{\max}$ — largest allowed input change

to obtain

$$d = \widehat{d}/\widehat{d}_{\max}, \quad u = \widehat{u}/\widehat{u}_{\max} \tag{3}$$

Scale $\widehat{y}$, $\widehat{e}$ and $\widehat{r}$ by

- $\widehat{e}_{\max}$ — largest allowed control error, or
- $\widehat{r}_{\max}$ — largest expected change in reference value

to (usually) obtain

$$y = \widehat{y}/\widehat{e}_{\max}, \quad r = \widehat{r}/\widehat{e}_{\max}, \quad e = \widehat{e}/\widehat{e}_{\max} \tag{4}$$

Scaling

MIMO :

$$d = D_d^{-1}\widehat{d}, \quad u = D_u^{-1}\widehat{u}, \quad y = D_e^{-1}\widehat{y}, \quad e = D_e^{-1}\widehat{e}, \quad r = D_e^{-1}\widehat{r} \qquad (5)$$

where $D_e = \widehat{e}_{\max}$, $D_u = \widehat{u}_{\max}$, $D_d = \widehat{d}_{\max}$ and $D_r = \widehat{r}_{\max}$ are diagonal scaling *matrices*. Substituting (5) into (2), we obtain

$$D_e y = \widehat{G} D_u u + \widehat{G}_d D_d d, \quad D_e e = D_e y - D_e r,$$

and introducing the scaled transfer functions

$$G = D_e^{-1}\widehat{G} D_u, \quad G_d = D_e^{-1}\widehat{G}_d D_d \qquad (6)$$

we can write the model in terms of the scaled variables, i.e.

$$y = Gu + G_d d; \quad e = y - r \qquad (7)$$

Often, we also write

$$\tilde{r} = \widehat{r}/\widehat{r}_{\max} = D_r^{-1}\widehat{r} \qquad (8)$$

so that

$$r = R\tilde{r} \quad \text{where} \quad R \stackrel{\Delta}{=} D_e^{-1} D_r = \widehat{r}_{\max}/\widehat{e}_{\max} \qquad (9)$$

Scaling

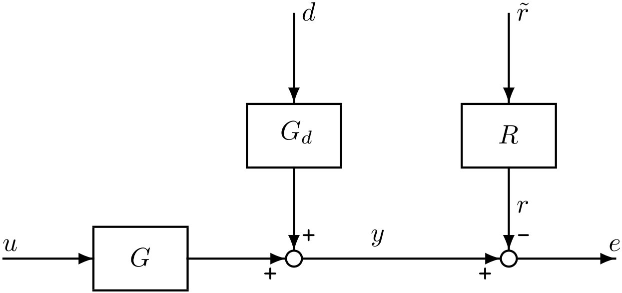
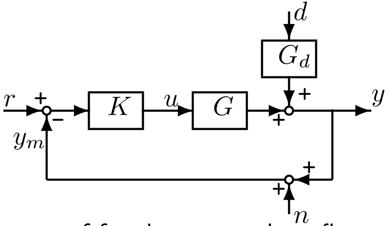


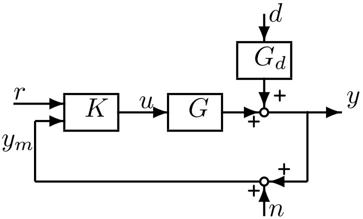
Figure 1: Model in terms of scaled variables

Objective:
For $|d(t)| \leq 1$ and $|\tilde{r}(t)| \leq 1$,
manipulate u with $|u(t)| \leq 1$
such that $|e(t)| = |y(t) - r(t)| \leq 1$.

Notation

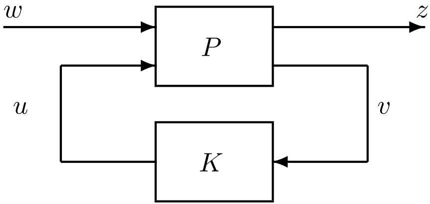


(a) One degree-of-freedom control configuration



(b) Two degrees-of-freedom control configuration

Notation



(c) General control configuration

Figure 2: Control configurations

Notation

Table 1: Nomenclature

K	controller, in whatever configuration. Sometimes broken down into parts. For example, in Figure 2(b), $K = [K_r \quad K_y]$ where K_r is a prefilter and K_y is the feedback controller.
Conventional configurations (Fig 2(a), 2(b)):	
G	plant model
G_d	disturbance model
r	reference inputs (commands, setpoints)
d	disturbances (process noise)
n	measurement noise
y	plant outputs. (include the variables to be controlled ("primary" outputs with reference values r) and possibly additional "secondary" measurements to improve control)
y_m	measured y
u	control signals (manipulated plant inputs)

Notation

General configuration (Fig 2(c)):

P generalized plant model. Includes G and G_d and the interconnection structure between the plant and the controller.
May also include weighting functions.

w exogenous inputs: commands, disturbances and noise

z exogenous outputs; "error" signals to be minimized, e.g. $y - r$

v controller inputs for the general configuration, e.g. commands, measured plant outputs, measured disturbances, etc. For the special case of a one degree-of-freedom controller with perfect measurements we have $v = r - y$.

u control signals

The Control Problem

Major difficulties:

Model (G, G_d) inaccurate $\Rightarrow$ RealPlant: $G_p = G + E$;

$E =$ "uncertainty" or "perturbation" (unknown but bounded)

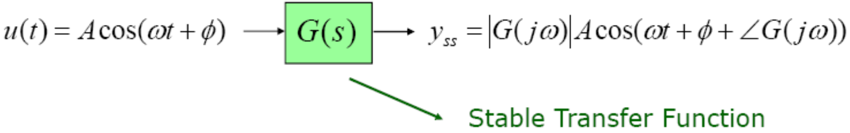
- **Nominal stability (NS)** : system is stable with no model uncertainty.
- **Nominal Performance (NP)** : system satisfies performance specifications with no model uncertainty.
- **Robust stability (RS)** : system stable for "all" perturbed plants
- **Robust performance (RP)** : system satisfies performance specifications for all perturbed plants

Transfer Functions

- Invaluable insights are obtained from simple frequency-dependent plots.
- Important concepts for feedback such as bandwidth and peaks of closed-loop transfer functions may be defined.
- $G(j\omega)$ gives the response to a sinusoidal input of frequency ω.
- A series of interconnected systems corresponds in the frequency domain to the multiplication of individual transfer functions, whereas in the time domain, the evaluation of convolution operations is required.
- Poles and zeros appear explicitly in factorized transfer functions.
- Uncertainty is more easily handled in frequency domain.

Frequency Response [2.1]:

- We use the *Frequency Response* to describe the response of the system to sinusoids of varying frequency.



Feedback control [2.2]:

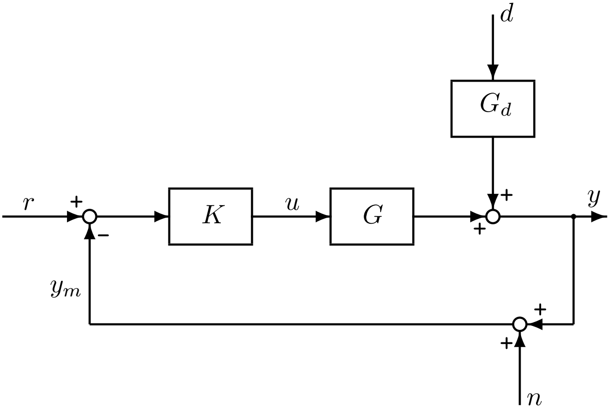


Figure 3: Block diagram of one degree-of-freedom feedback control system

Classical Feedback Control

The output can be written in this case as

$$y = GK(r - y - n) + G_d d$$

or

$$(I + GK)y = GKr + G_d d - GKn \tag{10}$$

Plant output:

$$y = \underbrace{(I + GK)^{-1}GK}_{T} r \tag{11}$$

$$+ \underbrace{(I + GK)^{-1}}_{S} G_d d \tag{12}$$

$$- \underbrace{(I + GK)^{-1}GK}_{T} n \tag{13}$$

Control error:

$$e = y - r = -Sr + SG_d d - Tn \tag{14}$$

Plant input:

$$u = KSr - KSG_d d - KSn \tag{15}$$

Classical Feedback Control

Note that:

$$L = GK \tag{16}$$

$$S = (I + GK)^{-1} = (I + L)^{-1} \tag{17}$$

$$T = (I + GK)^{-1}GK = (I + L)^{-1}L \tag{18}$$

$$S + T = I \tag{19}$$

Notation :

$L = GK$	loop transfer function
$S = (I + L)^{-1}$	sensitivity function
$T = (I + L)^{-1}L$	complementary sensitivity function

Closed-loop stability [2.3]:

- Root-locus
- Routh-Hurwitz criterion
- Nyquist criterion

Example (2.1/2.2/2.3) - Inverse-response Process: (lecture05a/b.m)

$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$

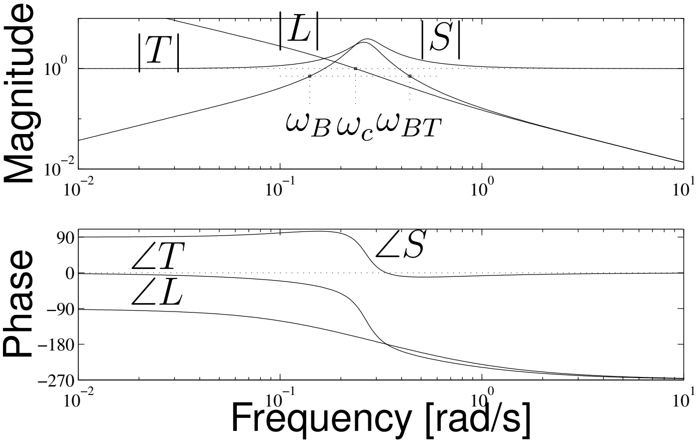


Figure 4: Bode magnitude and phase plots of $L = GK$, S and T when $G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$, and $K(s) = 1.136(1 + \frac{1}{12.7s})$

Closed-loop performance [2.4]:

Time domain performance

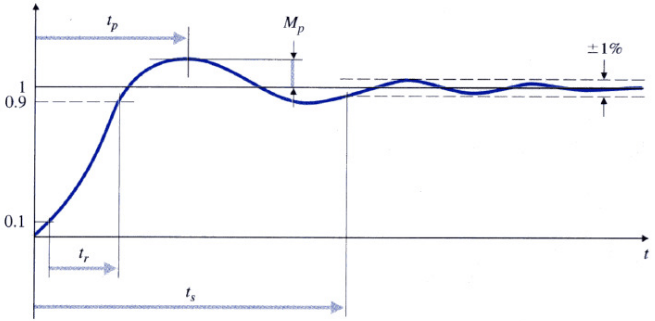


Figure 5: Rise time, Settling time, Overshoot, Decay Ratio, Steady-state Offset, Total Variation.

Frequency domain performance - Gain and phase margins

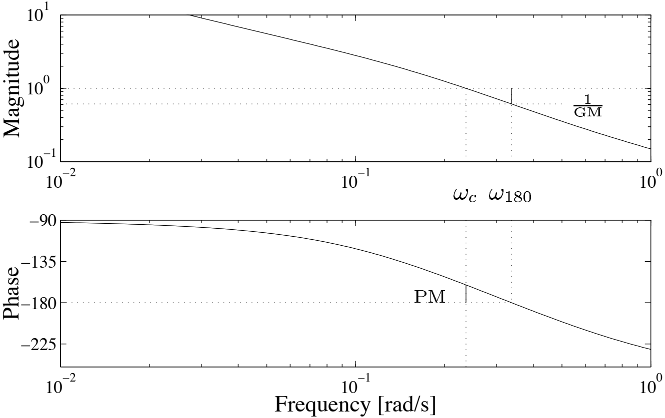


Figure 6: Typical Bode plot of $L(j\omega)$ with PM and GM indicated.

Classical Feedback Control

Frequency domain performance - Gain and phase margins

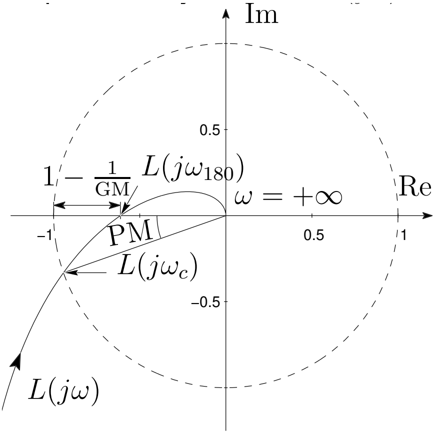


Figure 7: Typical Nyquist plot of $L(j\omega)$ for stable plant with PM and GM indicated. Closed-loop instability occurs if $L(j\omega)$ encircles the critical point -1

Frequency domain performance - Gain and phase margins

The *gain margin* is defined as

$$GM = 1/|L(j\omega_{180})|$$

where the *phase crossover frequency* ω_{180} is where the Nyquist curve of $L(j\omega)$ crosses the negative real axis between -1 and 0, that is

$$\angle L(j\omega_{180}) = -180^\circ$$

The GM is the factor by which the loop gain $|L(j\omega)|$ may be increased before the closed-loop system becomes unstable (i.e., before the gain at this frequency becomes 1). The GM is thus a direct safeguard against steady-state gain uncertainty (error). Typically we require $GM > 2$.

Frequency domain performance - Gain and phase margins

The *phase margin* is defined as

$$PM = \angle L(j\omega_c) + 180^\circ$$

where the *gain crossover frequency* ω_c is where $L(j\omega)$ first crosses 1 from above, that is

$$|L(j\omega_c)| = 1$$

The PM is the amount of negative phase (phase lag) we can add to $L(s)$ at frequency ω_c before the closed-loop system becomes unstable (i.e., before the phase at this frequency becomes -180°). Typically we require $PM > 30^\circ$. The PM is a direct safeguard against time delay uncertainty; the system becomes unstable if we add a time delay of

$$\theta_{max} = PM/\omega_c$$

.

Maximum peak criteria

Maximum peaks of sensitivity and complementary sensitivity functions:

$$M_S \stackrel{\triangle}{=} \max_{\omega} |S(j\omega)|; \quad M_T \stackrel{\triangle}{=} \max_{\omega} |T(j\omega)| \tag{20}$$

Note that $S + T \equiv 1$. Typically :

$$M_S \leq 2 \quad (6dB) \tag{21}$$

$$M_T < 1.25 \quad (2dB) \tag{22}$$

There is a close relationship between maximum peaks and gain/phase margins:

$$\text{GM} \geq \frac{M_S}{M_S - 1} \tag{23}$$

$$\text{PM} \geq 2 \arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} [\text{rad}] \tag{24}$$

For example, for $M_S = 2$ we are guaranteed

$$\text{GM} \geq 2 \text{ and } \text{PM} \geq 29.0^\circ.$$

Bandwidth and crossover frequency

- *Bandwidth* is defined as the frequency range $[\omega_1, \omega_2]$ over which control is "effective". Usually $\omega_1 = 0$, and then $\omega_2 = \omega_B$ is the bandwidth.

- The (closed-loop) bandwidth, ω_B, is the frequency where $|S(j\omega)|$ first crosses $1/\sqrt{2} = 0.707 (\approx -3 \text{ dB})$ from below.

- The bandwidth in terms of T, ω_{BT}, is the highest frequency at which $|T(j\omega)|$ crosses $1/\sqrt{2} = 0.707 (\approx -3 \text{ dB})$ from above. (Usually a poor indicator of performance).

- The *gain crossover frequency*, ω_c, is the frequency where $|L(j\omega_c)|$ first crosses 1 from above.For systems with PM $< 90^\circ$ we have

$$\omega_B < \omega_c < \omega_{BT} \tag{25}$$

Controller design [2.5]:

Three main approaches:

1. **Shaping of transfer functions.**
   - **Loop shaping.** Classical approach in which the magnitude of the open-loop transfer function, $L(j\omega)$, is shaped.
   - **Shaping of closed-loop transfer functions, such as S, T and KS $\Rightarrow$ $\mathcal{H}_\infty$** optimal control
2. **The signal-based approach.** One considers a particular disturbance or reference change and tries to optimize the closed-loop response $\Rightarrow$ Linear Quadratic Gaussian (LQG) control.
3. **Numerical optimization.** Multi-objective optimization to optimize directly the true objectives, such as rise times, stability margins, etc. Computationally difficult.

Loop shaping [2.6]:

Fundamentals of loop-shaping design

Shaping of open loop transfer function $L(j\omega)$:

$$e = -\underbrace{(I+L)^{-1}}_{S} r + \underbrace{(I+L)^{-1}}_{S} G_d d - \underbrace{(I+L)^{-1} L}_{T} n \tag{26}$$

Fundamental trade-offs:

1. Good disturbance rejection: L large.
2. Good command following: L large.
3. Mitigation of measurement noise on plant outputs: L small.
4. Small magnitude of input signals: K small and L small.

Classical Feedback Control

Fundamentals of loop-shaping design

Specifications for desired loop transfer function:

1. Gain crossover frequency, ω_c, where $|L(j\omega_c)| = 1$.

2. The shape of $L(j\omega)$, e.g. slope of $|L(j\omega)|$ in certain frequency ranges:

$$N = \frac{d\, ln\, |L|}{d\, ln\, w}$$

   Typically, a slope $N = -1$ (-20 dB/decade) around crossover, and a larger roll-off at higher frequencies. The desired slope at lower frequencies depends on the nature of the disturbance or reference signal.

3. The system type, defined as the number of pure integrators in $L(s)$.

Note:

- *$L(s)$ must contain at least one integrator for each integrator in $r(s)$.*
- *Slope and phase are dependent. For example:* $\angle \frac{1}{s^n} = -n\, \frac{\pi}{2}$

Fundamentals of loop-shaping design

Example (2.6) - Inverse-response Process: (lecture05c.m)

$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$

- The RHP zero cannot be cancelled by the controller. Therefore, L must contain the RHP zero of G.
- The RHP zero imposes a performance limitation. The crossover frequency must be $\omega_c < 0.5 * \omega_z$. In this case, $\omega_z = 0.5$.
- We require the system to have one integrator (type 1 system).
- CONCLUSION: L must have a slope of -20dB/dec at low frequencies and then roll off with a higher slope at frequencies beyond ω_z.

$$L(s) = K_c \frac{3(-2s+1)}{s(2s+1)(0.33s+1)} \Rightarrow K(s) = K_c \frac{(10s+1)(5s+1)}{s(2s+1)(0.33s+1)}$$

Inverse-based controller [2.6.3]

Note: $L(s)$ must contain all RHP-zeros of $G(s)$, but otherwise the specified $L(s)$ is independent of $G(s)$. This suggest the following idea for minimum phase plants:

$$L(s) = \frac{\omega_c}{s} \tag{27}$$

$$K(s) = \frac{\omega_c}{s} G^{-1}(s) \tag{28}$$

The controller inverts plant and adds integrator $(1/s)$. We select a loop shape that has a slope of $N = -1$ throughout the frequency range! There are at least two good reasons for why this inverse-based controller may not be a good choice:

- The controller will not be realizable if $G(s)$ has a pole excess of two or larger, and may in any case yield large input signals. These problems may be partly fixed by adding high-frequency dynamics to the controller.
- The L slope of $N = -1$ throughout the frequency range is *not* generally desirable, unless references and disturbances affect the outputs as steps.

Example (2.7) - Disturbance Process: (lecture05d.m)

$$\boxed{G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}, \quad G_d(s) = \frac{100}{10s+1}} \tag{29}$$

Objectives are:

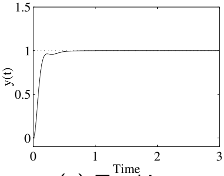
1. Command tracking: rise time (to reach 90% of the final value) less than 0.3 s and overshoot less than 5%.
2. Disturbance rejection: response to unit step disturbance should stay within the range $[-1, 1]$ at all times, and should return to 0 as quickly as possible ($|y(t)|$ should at least be less than 0.1 after 3 s).
3. Input constraints: $u(t)$ should remain within $[-1, 1]$

Analysis. $G_d(0) = 100!!!$ $|G_d(j\omega)|$ remains larger than 1 up to $\omega_d \approx 10$ rad/s $\Rightarrow$ feedback needed up to $\omega_d \approx 10$ rad/s $\Rightarrow$ $\underline{\omega_c \approx 10 \text{ rad/s}}$. No larger than this (noise sensitivity & stability problems associated with high gain feedback).

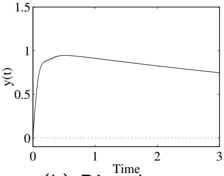
Inverse-based controller design.

$$K_0(s) = \frac{\omega_c}{s}\frac{10s+1}{200}(0.05s+1)^2 \approx \frac{\omega_c}{s}\frac{10s+1}{200}\frac{0.1s+1}{0.01s+1},$$

$$L_0(s) = \frac{\omega_c}{s}\frac{0.1s+1}{(0.05s+1)^2(0.01s+1)},\ \omega_c = 10 \tag{30}$$



(a) Tracking



(b) Disturbance

Figure 8: Responses with "inverse-based" controller $K_0(s)$ for the disturbance process. Note poor disturbance response in spite of good reference response.

Loop shaping for disturbance rejection [2.6.4]

$$e = y = SG_d d, \tag{31}$$

to achieve $|e(\omega)| \leq 1$ for $|d(\omega)| = 1$ (the worst-case disturbance) we require $|SG_d(j\omega)| \leq 1, \forall\omega$, or

$$|1 + L| \geq |G_d| \quad \forall\omega \tag{32}$$

At frequencies where $|G_d| > 1$, this is approximately equivalent to:

$$|L| \geq |G_d| \quad \forall\omega \tag{33}$$

Initial guess:

$$|L_{\min}| \approx |G_d| \tag{34}$$

or:

$$|K_{\min}| \approx |G^{-1}G_d| \tag{35}$$

Controller contains the model of the disturbance!

In addition, to improve low-frequency performance (e.g. to get zero steady-state offset), we often add integral action at low frequencies

$$|K| = |\frac{s+\omega_I}{s}||G^{-1}G_d| \tag{36}$$

Summary:

- Controller contains the dynamics (G_d) of the disturbance and inverts the dynamics (G) of the inputs.
- For disturbance at plant output, $G_d = 1$, we get $|K_{\min}| = |G^{-1}|$
  - So, an inverse-based design provides the best trade-off between performance (disturbance rejection) and minimum use of feedback
- For disturbances at plant input we have $G_d = G$ and we get $|K_{\min}| = 1$
  - So, a simple proportional controller with unit gain yields a good trade-off between output performance and input usage
- *Note that a reference change may be viewed as a disturbance affecting the output. A maximum reference change* $r = R$ *may be viewed as a disturbance* $d = 1$ *with* $G_d(s) = -R$ *where* R *is usually a constant. This explains why* $K = G^{-1}$ *(inverse-based controller) yields good step-reference response.*

Classical Feedback Control

In addition to satisfying $|L| \approx |G_d|$ at frequencies around crossover, the desired loop-shape $L(s)$ may be modified as follows:

1. Around crossover make slope N of $|L|$ to be about -1 for transient behaviour with acceptable gain and phase margins.
2. Increase the loop gain at low frequencies to improve the settling time and reduce the steady-state offset $\rightarrow$ add an integrator.
3. Let $L(s)$ roll off faster at higher frequencies (beyond the bandwidth) in order to reduce the use of manipulated inputs, to make the controller realizable and to reduce the effects of noise.

Moreover:

- The (phase) of $L(s)$ must be selected s.t. the closed-loop system is stable
- When making $|L| \approx |G_d|$, replace $G_d(s)$ by the corresponding minimum-phase transfer function with the same magnitude because time delays and RHP-zeros will impose undesirable limitations on feedback
- Any any time delays or RHP-zeros in $G(s)$ must be included in $L(s)$ because RHP pole-zero cancellations yield internal instability

Example (2.8): Loop-shaping for Disturbance Process (lecture05e.m)

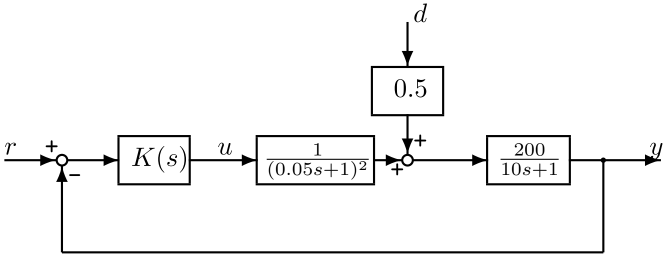


Figure 9: Block diagram representation of the disturbance process in (37)

$$\boxed{G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}, \quad G_d(s) = \frac{100}{10s+1}} \tag{37}$$

- G and G_d share the same dominating dynamics as represented by the term $200/(10s+1)$

Step 1. Initial design. $K(s) = G^{-1}G_d = 0.5(0.05s+1)^2$.
Make proper ($(0.05s+1)^2 \approx 1$ until $\omega = 20 rad/sec$):

$$K_1(s) = 0.5 \tag{38}$$

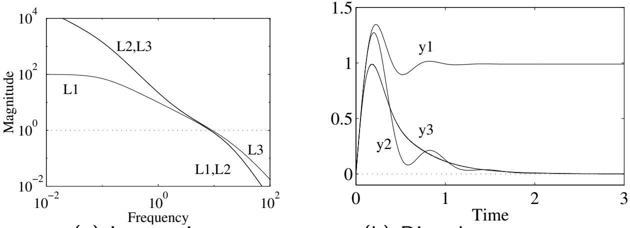
$\Longrightarrow$ offset (no integrator)! Similar to more complex inverse-based controller.
Step 2. More gain at low frequency. To get integral action multiply the controller by the term $\frac{s+\omega_I}{s}$. For $\omega_I = 0.2\omega_c$ the phase contribution from $\frac{s+\omega_I}{s}$ is $\arctan(1/0.2) - 90° = -11°$ at ω_c. For $\omega_c \approx 10$ rad/s, select controller:

$$K_2(s) = 0.5\frac{s+2}{s} \tag{39}$$

$\Longrightarrow$ response exceeds 1, oscillatory, small phase margin ($|y(t)| < 0.1$s at $t = 3$s)
Step 3. High-frequency correction. Supplement with "derivative action" by multiplying $K_2(s)$ by lead-lag term effective over one decade starting at 20 rad/s:

$$K_3(s) = 0.5\frac{s+2}{s}\,\frac{0.05s+1}{0.005s+1} \tag{40}$$

$\Longrightarrow$ poor reference tracking (simulation) ($PM = 51°$)



(a) Loop gains (b) Disturbance responses

Figure 10: Loop shapes and disturbance responses for controllers K_1, K_2 and K_3 for the disturbance process

Two Degrees of Freedom (DoF) Design [2.6.5]

In order to meet <u>both</u> regulator and tracking performance use K_r (= "prefilter"):

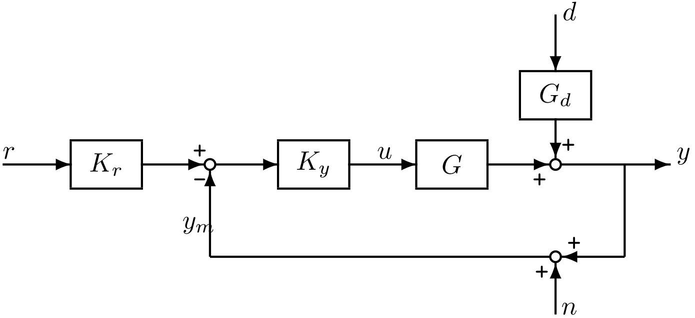


Figure 11: Two degrees-of-freedom controller

Reference Tracking: $K \propto \frac{G^{-1}}{s}$

Disturbance Rejection: $K \propto \frac{G^{-1}G_d}{s}$

Classical Feedback Control

Idea:

- Design K_y for Disturbance Rejection
- $T = L(I+L)^{-1}$ with $L = GK_y$
- Desired $y = T_{ref}r$ (Reference Tracking)

$$\Longrightarrow K_r = T^{-1}T_{ref} \tag{41}$$

Remark:
Practical choice of prefilter is the lead-lag network

$$K_r(s) = \frac{\tau_{\text{lead}}s+1}{\tau_{\text{lag}}s+1} \tag{42}$$

$\tau_{\text{lead}} > \tau_{\text{lag}}$ to speed up the response, and $\tau_{\text{lead}} < \tau_{\text{lag}}$ to slow down the response.

Example (2.9): Two DoF Design for Disturbance Process (lecture05e.m)

$K_y = K_3$. Approximate response by inpection of y_3:

$$T(s) \approx \frac{1.5}{0.1s+1} - \frac{0.5}{0.5s+1} = \frac{(0.7s+1)}{(0.1s+1)(0.5s+1)}$$

which yields (with $T_{ref} = 1/(0.1s+1)$):

$$K_r(s) = T^{-1}T_{ref} = \frac{0.5s+1}{0.7s+1}$$

.

By closed-loop simulations:

$$K_{r3}(s) = \frac{0.5s+1}{0.65s+1} \cdot \frac{1}{0.03s+1} \quad (43)$$

where $1/(0.03s+1)$ included to avoid initial peaking of input signal $u(t)$ above 1.

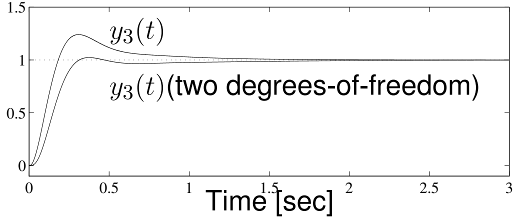


Figure 12: Tracking responses with the one degree-of-freedom controller (K_3) and the two degrees-of-freedom controller (K_3, K_{r3}) for the disturbance process

Closed-loop Shaping [2.7]:

Why?

We are interested in S and T:

$$\begin{array}{lcll} |L(j\omega)| \gg 1 & \Rightarrow & S \approx L^{-1}; & T \approx 1 \\ |L(j\omega)| \ll 1 & \Rightarrow & S \approx 1; & T \approx L \end{array}$$

but in the crossover region where $|L(j\omega)|$ is close to 1, one cannot infer anything about S and T from $|L(j\omega)|$.

Alternative: Directly shape the magnitudes of closed-loop $S(s)$ and $T(s)$.

The Term $\mathcal{H}_\infty$[2.7.1]

The $\mathcal{H}_\infty$ norm of a stable scalar transfer function $f(s)$ is simply the peak value of $|f(j\omega)|$ as a function of frequency, that is,

$$\|f(s)\|_\infty \stackrel{\Delta}{=} \max_\omega |f(j\omega)| \tag{44}$$

The symbol ∞ comes from:

$$\max_\omega |f(j\omega)| = \lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |f(j\omega)|^p d\omega \right)^{1/p}$$

The symbol $\mathcal{H}$ stands for "Hardy space", and $\mathcal{H}_\infty$ is the set of transfer functions with bounded ∞-norm, which is simply the set of *stable and proper* transfer functions.

Weighted Sensitivity [2.7.2]

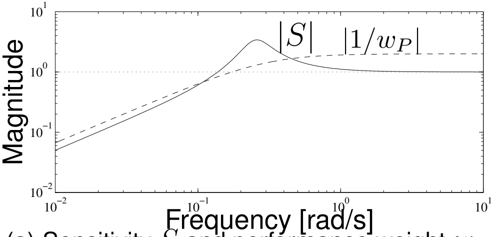
Typical specifications in terms of S:

1. Minimum bandwidth frequency ω_B^*.
2. Maximum tracking error at selected frequencies.
3. System type, or alternatively the maximum steady-state tracking error, A.
4. Shape of S over selected frequency ranges.
5. Maximum peak magnitude of S, $\|S(j\omega)\|_\infty \leq M$.

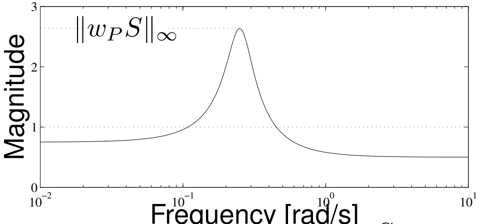
Specifications may be captured by an upper bound, $1/|w_P(s)|$, on $\|S\|$.

$$|S(j\omega)| < 1/|w_P(j\omega)|,\ \forall\omega \quad \Leftrightarrow \quad |w_P S| < 1,\ \forall\omega \quad \Leftrightarrow \quad \boxed{\|w_P S\|_\infty < 1} \tag{45}$$

Classical Feedback Control



(a) Sensitivity S and performance weight w_P



(b) Weighted sensitivity $w_P S$

Figure 13: $|S|$ exceeds its bound $1/|w_P| \Rightarrow \|w_P S\|_\infty > 1$

Typical performance weight:

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \tag{46}$$

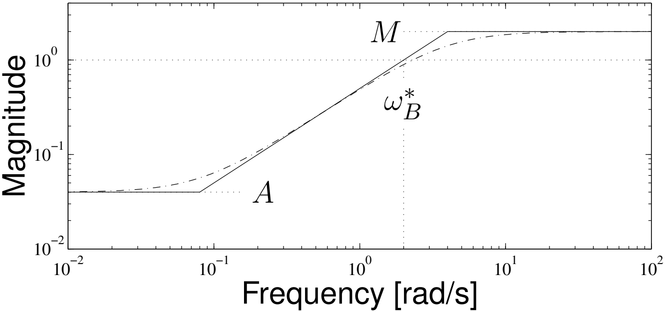


Figure 14: Inverse of performance weight. Exact & asymptotic plot of $1/|w_P(j\omega)|$ in (46)

Classical Feedback Control

To get a steeper slope for L (and S) below the bandwidth:

$$\boxed{w_P(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}} \tag{47}$$

Mixed Sensitivity [2.7.3]

In order to enforce specifications on other transfer functions:

$$\|N\|_\infty = \max_\omega \bar{\sigma}(N(j\omega)) < 1; \quad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u K S \end{bmatrix} \tag{48}$$

For SISO systems, N is a vector and the maximun singular value $\bar{\sigma}(N)$ is the usual Euclidean vector norm:

$$\bar{\sigma}(N) = \sqrt{|w_P S|^2 + |w_T T|^2 + |w_u K S|^2} \tag{49}$$

The $\mathcal{H}_\infty$ optimal controller is obtained from

$$\min_K \|N(K)\|_\infty \tag{50}$$

Example (2.11): $\mathcal{H}_\infty$ Design for Disturbance Process (lecture05f1/2.m)

Consider the plant in (29), and an $\mathcal{H}_\infty$ mixed sensitivity S/KS design in which

$$N = \begin{bmatrix} w_P S \\ w_u KS \end{bmatrix} \tag{51}$$

Selected $w_u = 1$ and

$$w_{P1}(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}; \quad M = 1.5,\ \omega_B^* = 10, \quad A = 10^{-4} \tag{52}$$

$\Longrightarrow$ poor disturbance response

To get higher gains at low frequencies:

$$w_{P2}(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}, \quad M = 1.5, \omega_B^* = 10, A = 10^{-4} \tag{53}$$

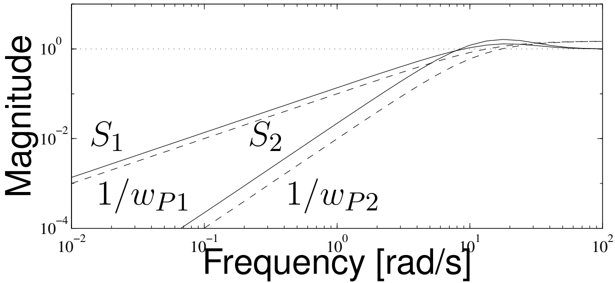


Figure 15: Inverse of performance weight (dashed line) and resulting sensitivity function (solid line) for two $\mathcal{H}_\infty$ designs (1 and 2) for the disturbance process

Classical Feedback Control

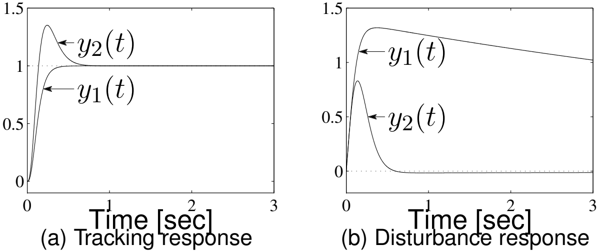


Figure 16: Closed-loop step responses for two alternative $\mathcal{H}_\infty$ designs (1 and 2) for the disturbance process