System Identification and Robust Control

Lecture 3: Nonparametric Identification

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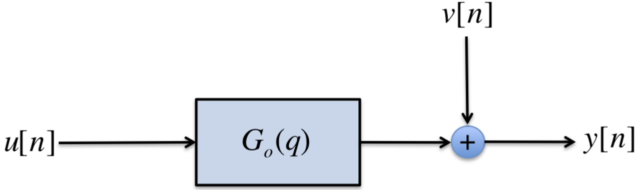
Nonparametric Identification

- Time-Domain Methods
  - Impulse response
  - Step response
  - Correlation analysis (time)
- Frequency-Domain Methods
  - Sinusoidal excitation
  - Correlation analysis (frequency)
  - Fourier Analysis
  - Spectral Analysis

Nonparametric Identification

- Causal, LTI systems:

$$y[n] = g_o[n] * u[n] + v[n] = \sum_{k=0}^{\infty} g_o[k]u[n-k] + v[n] = G_o(q)u[n] + v[n] \quad (1)$$



  – Time-Domain Methods $\rightarrow$ Estimation of g_o

  – Frequency-Domain Methods $\rightarrow$ Estimation of $G_o(e^{i\omega})$

Time - Impulse Response

Let us consider the system described by (1) subject to a pulse input

$$u[n] = \begin{cases} \alpha, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{2}$$

Then the output will be

$$y[n] = \sum_{k=0}^{\infty} g_o[k]u[n-k] + v[n] = \alpha g_o[n] + v[n] \tag{3}$$

It is then possible to estimate the impulse-response coefficients $g_o[n]$ as

$$\widehat{g}[n] = \frac{y[n]}{\alpha} \tag{4}$$

Time - Impulse Response

The estimate error is given by

$$|g_o[n] - \widehat{g}[n]| = \frac{|v[n]|}{\alpha}$$

- The error is small if the noise level is low or if $\alpha >> 1$
- Physical processes do not allow pulse inputs of high amplitude ($\alpha >> 1$)
- Pulse inputs of high amplitude ($\alpha >> 1$) can make the system exhibit nonlinear effects (the linear assumption will not longer hold)
- Estimates of impulse-response coefficients would suffer from large errors in most practical applications

Time - Step Response

Let us consider the system described by (1) subject to a step input

$$u[n] = \begin{cases} \alpha, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{5}$$

Then the output will be

$$y[n] = \sum_{k=0}^{\infty} g_o[k]u[n-k] + v[n] = \sum_{k=0}^{n} \alpha g_o[k] + v[n] \tag{6}$$

It is then possible to estimate the impulse-response coefficients $g_o[n]$ as

$$\widehat{g}[n] = \frac{y[n] - y[n-1]}{\alpha} \tag{7}$$

Time - Step Response

The estimate error is given by

$$|g_o[n] - \widehat{g}[n]| = \frac{|v[n] - v[n-1]|}{\alpha}$$

- The error is small if the noise level is low or if $\alpha >> 1$
- Physical processes do not allow step inputs of high amplitude ($\alpha >> 1$). Even if they are allowed, linearity assumption may not longer be valid.
- Estimates of impulse-response coefficients would suffer from large errors in most practical applications
- The method can be very useful in determining basic control-related characteristics such as delay times, static gains and time constants.

Time - Correlation Analysis

Let us consider the system described by (1) subject to a quasi-stationary input sequence with

$$R_{uu}[\tau] = E\{u[n]u[n-\tau]\} \tag{8}$$

and independent of the noise, i.e.,

$$E\{u[n]v[n-\tau]\} \equiv 0 \tag{9}$$

Note that this requires open-loop operation!

The cross-correlation between input $u[n]$ and output $y[n]$ is

$$\begin{aligned} R_{yu}[\tau] &= E\{y[n]u[n-\tau]\} \\ &= E\{(G_o(q)u[n] + v[n])u[n-\tau]\} \\ &= E\{G_o(q)u[n]u[n-\tau]\} + E\{v[n]u[n-\tau]\} \\ &= E\{G_o(q)u[n]u[n-\tau]\} \end{aligned}$$

Time - Correlation Analysis

Then,

$$\begin{aligned} R_{yu}[\tau] &= E\{G_o(q)u[n]u[n-\tau]\} \\ &= E\left\{\sum_{k=0}^{\infty} g_o[k]u[n-k]u[n-\tau]\right\} \\ &= \sum_{k=0}^{\infty} g_o[k]E\{u[n-k]u[n-\tau]\} \\ &= \sum_{k=0}^{\infty} g_o[k]R_{uu}[\tau-k] \\ &= g_o[\tau] * R_{uu}[\tau] \end{aligned}$$

Time - Correlation Analysis

The estimators for the auto-correlation and cross-correlation are defined as

$$\widehat{R}_{uu}[\tau] = \frac{1}{N}\sum_{k=0}^{N-1} u[k]u[k-\tau] = \frac{1}{N}\sum_{k=\tau}^{N-1} u[k]u[k-\tau], \quad (10)$$

$$\widehat{R}_{uv}[\tau] = \frac{1}{N}\sum_{k=0}^{N-1} u[k]v[k-\tau] = \frac{1}{N}\sum_{k=\tau}^{N-1} u[k]v[k-\tau]. \quad (11)$$

Both estimators are asymptotically unbiased

$$\lim_{N\to\infty} E\{\widehat{R}_{uu}[\tau]\} = R_{uu}[\tau], \quad \lim_{N\to\infty} E\{\widehat{R}_{uv}[\tau]\} = R_{uv}[\tau]$$

and in addition it can be showed that

$$E\{\widehat{R}[\tau]\} = \frac{N-|\tau|}{N} R[\tau]. \quad (12)$$

Time - Correlation Analysis

$$R_{yu}[\tau]=E\{y[n]u[n-\tau]\}=\sum_{k=0}^{\infty} g_o[k]R_{uu}[\tau-k]=g_o[\tau]*R_{uu}[\tau] \tag{13}$$

- Input is white noise:

$$R_{uu}[\tau] = E\{u[n]u[n-\tau]\} = \alpha\delta[\tau] \tag{14}$$

Note that

$$R_{uu}[0] = E\{u[n]^2\} = \alpha \tag{15}$$

Then,

$$R_{yu}[\tau] = \sum_{k=0}^{\infty} g_o[k]\alpha\delta[\tau-k] = g_o[\tau]*\alpha\delta[\tau] = \alpha g_o[\tau] \tag{16}$$

We determine the impulse-response coefficients $g_o[\tau]$ as

$$g_o[\tau] = \frac{R_{yu}[\tau]}{\alpha} = \frac{R_{yu}[\tau]}{R_{uu}[0]} \tag{17}$$

Time - Correlation Analysis

- Taking into account the estimators for the auto-correlation and cross-correlation defined in (10) and (11) respectively, it is then possible to estimate the impulse-response coefficients $g_o[\tau]$ as

$$\widehat{g}[\tau] = \frac{\widehat{R}_{yu}[\tau]}{\widehat{R}_{uu}[0]} \tag{18}$$

where

$$\widehat{R}_{uu}[0] = \frac{1}{N}\sum_{k=0}^{N-1} u[k]^2, \quad \widehat{R}_{yu}[\tau] = \frac{1}{N}\sum_{k=\tau}^{N-1} y[k]u[k-\tau]. \tag{19}$$

  - $E\{\widehat{g}[\tau]\} \to g_o[\tau]$ as $N \to \infty$
  - Covariance of the estimation is proportional to $\frac{1}{N}$

Time - Correlation Analysis

$$R_{yu}[\tau]=E\{y[n]u[n-\tau]\}=\sum_{k=0}^{\infty} g_o[k]R_{uu}[\tau-k]=g_o[\tau]*R_{uu}[\tau] \tag{20}$$

- Input is NOT white noise.
- Taking into account the estimators for the auto-correlation and cross-correlation defined in (10) and (11) respectively, it is then possible to estimate the impulse-response coefficients $g_o[\tau]$ from the relationship

$$\widehat{R}_{yu}[\tau] = \sum_{k=0}^{M-1} \widehat{g}[k]\widehat{R}_{uu}[\tau-k] \tag{21}$$

$$\begin{bmatrix} \widehat{R}_{yu}[0] \\ \widehat{R}_{yu}[1] \\ \vdots \\ \widehat{R}_{yu}[M-1] \end{bmatrix} = \begin{bmatrix} \widehat{R}_{uu}[0] & \widehat{R}_{uu}[-1] & \cdots & \widehat{R}_{uu}[-(M-1)] \\ \widehat{R}_{uu}[1] & \widehat{R}_{uu}[0] & \cdots & \widehat{R}_{uu}[-(M-2)] \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{R}_{uu}[M-1] & \widehat{R}_{uu}[M-2] & \cdots & \widehat{R}_{uu}[0] \end{bmatrix} \begin{bmatrix} \widehat{g}[0] \\ \widehat{g}[1] \\ \vdots \\ \widehat{g}[M-1] \end{bmatrix}$$

Time - Correlation Analysis

- Notice that $\widehat{R}_{uu}[\tau] = \widehat{R}_{uu}[-\tau]$. The estimates are given by

$$\begin{bmatrix} \widehat{g}[0] \\ \widehat{g}[1] \\ \vdots \\ \widehat{g}[M-1] \end{bmatrix} = \begin{bmatrix} \widehat{R}_{uu}[0] & \widehat{R}_{uu}[-1] & \cdots & \widehat{R}_{uu}[-(M-1)] \\ \widehat{R}_{uu}[1] & \widehat{R}_{uu}[0] & \cdots & \widehat{R}_{uu}[-(M-2)] \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{R}_{uu}[M-1] & \widehat{R}_{uu}[M-2] & \cdots & \widehat{R}_{uu}[0] \end{bmatrix}^{-1} \begin{bmatrix} \widehat{R}_{yu}[0] \\ \widehat{R}_{yu}[1] \\ \vdots \\ \widehat{R}_{yu}[M-1] \end{bmatrix}$$

  - Unique solution $\iff$ Persistency of Excitation

  - Same estimate regardless of the noise spectrum

Frequency - Sinusoidal Excitation

- Let us have a Discrete-Time LTI system described by

$$y[n] = G(q)u[n] + v[n], \tag{22}$$

where $v[n]$ represents a noise sequence. If the input sequence $u[n]$ is given by

$$u[n] = A\cos(\omega_o n). \tag{23}$$

We already showed that the output will be

$$y[n] = A\left|G(e^{j\omega_o})\right|\cos(\omega_o n + \arg[G(e^{j\omega_o})]) + v[n]. \tag{24}$$

Recall that $\cos(\omega_o n) = (e^{j\omega_o n} + e^{-j\omega_o n})/2$.

- With input (23) determine the amplitude and phase shift of resulting output cosine signal (24). Calculate an estimate $\widehat{G}(e^{j\omega_o})$ based on that information. Repeat for a number of frequencies ω_o in the frequency band of interest.
- This is known as *frequency analysis* and is a simple method for obtaining detailed information about a linear system.

Frequency - Sinusoidal Excitation

- In presence of noise $v[n]$ it may be difficult to determine $\left|G(e^{j\omega_o})\right|$ and $\arg[G(e^{j\omega_o})]$. A good approach to deal with this problem is Correlation Analysis. Since the component of $y[n]$ of interest is a cosine function of known frequency, it is possible to correlate it out from the noise.

- Given the sums

$$I_c(N) = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\cos(\omega_o n), \quad I_s(N) = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\sin(\omega_o n), \tag{25}$$

we can insert (24) in (25) to obtain:

Frequency - Correlation Analysis

$$
\begin{aligned}
I_c(N) &= \frac{1}{N}\sum_{n=0}^{N-1} A\left|G(e^{j\omega_o})\right|\cos(\omega_o n + \arg[G(e^{j\omega_o})])\cos(\omega_o n) \\
&\quad + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\cos(\omega_o n) \\
&= A\left|G(e^{j\omega_o})\right|\frac{1}{2}\frac{1}{N}\sum_{n=0}^{N-1}\Big[\cos(2\omega_o n + \arg[G(e^{j\omega_o})]) \\
&\quad + \cos(\arg[G(e^{j\omega_o})])\Big] + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\cos(\omega_o n) \\
&= \frac{A}{2}\left|G(e^{j\omega_o})\right|\cos(\arg[G(e^{j\omega_o})]) \\
&\quad + \frac{A}{2}\left|G(e^{j\omega_o})\right|\frac{1}{N}\sum_{n=0}^{N-1}\cos(2\omega_o n + \arg[G(e^{j\omega_o})]) \\
&\quad + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\cos(\omega_o n)
\end{aligned}
$$

Frequency - Correlation Analysis

$$
\begin{aligned}
I_s(N) \quad = \quad & -\frac{A}{2}\left|G(e^{j\omega_o})\right| \sin(\arg[G(e^{j\omega_o})]) \\
& + \frac{A}{2}\left|G(e^{j\omega_o})\right| \frac{1}{N}\sum_{n=0}^{N-1} \sin(2\omega_o n + \arg[G(e^{j\omega_o})]) \\
& + \frac{1}{N}\sum_{n=0}^{N-1} v[n]\sin(\omega_o n).
\end{aligned}
$$

Frequency - Correlation Analysis

- When $N \to \infty$, the second and third terms in the expressions for $I_c(N)$ and $I_s(N)$ tend to zero. We finally obtain

$$I_c(N) = \frac{A}{2} \left| G(e^{j\omega_o}) \right| \cos(\arg[G(e^{j\omega_o})]) \tag{26}$$

$$I_s(N) = -\frac{A}{2} \left| G(e^{j\omega_o}) \right| \sin(\arg[G(e^{j\omega_o})]). \tag{27}$$

- These two expressions suggest the following estimators for the Frequency Response:

$$\left| \widehat{G}(e^{j\omega_o}) \right| = \frac{\sqrt{I_c(N)^2 + I_s(N)^2}}{A/2} \tag{28}$$

$$\widehat{\arg}[G(e^{j\omega_o})] = -\arctan \frac{I_s(N)}{I_c(N)}. \tag{29}$$

- The Discrete Fourier Transform of the output sequence $y[n]$ is given by

$$Y(\omega) = \sum_{n=0}^{N-1} y[n] e^{-j\omega n}, \tag{30}$$

with $\omega = \frac{2\pi}{N} k \, (0 \leq k \leq N-1; 0 \leq \omega < 2\pi)$.

- Comparing this expression with (25) we can write

$$I_c(N) - jI_s(N) = \frac{1}{N} Y(\omega_o). \tag{31}$$

- The Discrete Fourier Transform of the input sequence $u[n]$ is computed as

$$U(\omega) = \sum_{n=0}^{N-1} u[n] e^{-j\omega n} = \sum_{n=0}^{N-1} A \frac{e^{j\omega_o n} + e^{-j\omega_o n}}{2} e^{-j\omega n}, \tag{32}$$

with $\omega = \frac{2\pi}{N} k \, (0 \leq k \leq N-1; 0 \leq \omega < 2\pi)$.

- Exploiting the periodicity property of the complex exponential function, this results in

$$U(\omega) = \begin{cases} N\frac{A}{2} & \text{for } \omega = \omega_o \text{ if } \omega_o = \frac{2\pi}{N}k \\ & \text{for some integer } 0 \leq k \leq N-1, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

- It is straightforward now to show that

$$\left|\widehat{G}(e^{j\omega_o})\right| = \left|\frac{Y(\omega_o)}{U(\omega_o)}\right| \quad (34)$$

$$\widehat{\arg}[G(e^{j\omega_o})] = \arg\left[\frac{Y(\omega_o)}{U(\omega_o)}\right], \quad (35)$$

  which means that an estimation of the Frequency Response at the frequency of the input signal can be computed based on the Discrete Fourier Transform of the input and output sequences:

$$\widehat{G}(e^{j\omega_o}) = \frac{Y(\omega_o)}{U(\omega_o)}. \quad (36)$$

- Note that this result is consistent with the DFT-filtering Theorem (Equation (58) - Lecture 2), where $R(\omega) = 0$ for periodic inputs.

Frequency - ETFE

- We found that equation (36) corresponds to frequency analysis with a single sinusoid of frequency ω_o as input.
- In a linear system, different frequencies pass through the system independently of each other (thanks to linearity!!!). It is therefore quite natural to extend the frequency analysis estimate (36) to the case of multi-frequency inputs.
- We introduce the following estimate of the transfer function:

$$\widehat{\widehat{G}}(e^{j\omega}) = \frac{Y(\omega)}{U(\omega)}. \tag{37}$$

- We call the estimate (37) the *empirical transfer-function estimate* (ETFE). We assume that $U(\omega) \neq 0$. If this does not hold for some frequencies, we consider the ETFE as undefined for those frequencies.

Frequency - ETFE

- Let us consider both $y[n]$ and $u[n]$ sequences of length N.
- Then, the ETFE is a complex sequence of length N providing an estimate of the transfer function at frequencies $\omega = 2\pi k/N$, $k = 0, \ldots, N-1$.
- However, since $y[n]$ and $u[n]$ are real, we have

$$\widehat{\widehat{G}}(e^{j2\pi k/N}) = \widehat{\widehat{G}}^{*}(e^{j2\pi(N-k)/N}).$$

- Therefore the ETFE consists of $N/2$ essential points.
- For the following lemmas, Let $U(\omega)$ and $V(\omega)$ be defined by

$$U(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u[n] e^{-j\omega n}, \quad V(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} v[n] e^{-j\omega n}$$

Frequency - ETFE

Lemma 1: Consider a strictly stable system

$$y[n] = G_o(q)u[n] + v[n]$$

with a disturbance $v[n]$ being a stationary stochastic process with spectrum $\Phi_{vv}(\omega)$ and covariance function $R_{vv}[\tau]$ subject to

$$\sum_{-\infty}^{\infty} |\tau R_{vv}[\tau]| < \infty$$

Let $u[n]$ be independent of $v[n]$ and assume that $|u[n]| \leq C \; \forall n$. Then,

$$E\{\widehat{\widehat{G}}(e^{j\omega})\} = G_o(e^{j\omega}) + \frac{\rho_1(N)}{U(\omega)} \tag{38}$$

where $|\rho_1(N)| \leq \frac{C_1}{\sqrt{N}}$.

And,

$$E\left\{\left(\widehat{\widehat{G}}(e^{j\omega})-G_o(e^{j\omega})\right)\left(\widehat{\widehat{G}}(e^{-j\xi})-G_o(e^{-j\xi})\right)\right\} \tag{39}$$

$$= \begin{cases} \frac{1}{|U(\omega)|^2}[\Phi_{vv}(\omega)+\rho_2(N)] & \text{if } \xi=\omega \\ \frac{\rho_2(N)}{U(\omega)U(-\xi)} & \text{if } |\xi-\omega|=\frac{2\pi k}{N},\ k=1,2,\ldots,N-1 \end{cases}$$

where $|\rho_2(N)| \leq \frac{C_2}{\sqrt{N}}$. The constants are given by

$$\begin{aligned} C_1 &= \left(2\sum_{k=1}^{\infty}|kg_o[k]|\right)\max|u[n]| \\ C_2 &= C_1^2+\sum_{-\infty}^{\infty}|\tau R_{vv}[\tau]| \end{aligned}$$

If $u[n]$ periodic $\Rightarrow \rho_1(N)=0$ at $\omega=2\pi k/N \Rightarrow C_1=0$.

Lemma 2: Let $v[n]$ be given by

$$v[n] = H(q)e[n]$$

where $e[n]$ is a white noise sequence with variance λ and fourth moment μ^2 and H is a strictly stable filter. Let $\Phi_{vv}(\omega)$ be the spectrum of $v[n]$. Then,

$$E\left\{|V(\omega)|^2\right\} = \Phi_{vv}(\omega) + \rho_3(N) \tag{40}$$

$$E\left\{\left(|V(\omega)|^2 - \Phi_{vv}(\omega)\right)\left(|V(\xi)|^2 - \Phi_{vv}(\xi)\right)\right\} \tag{41}$$

$$= \begin{cases} [\Phi_{vv}(\omega)]^2 + \rho_4(N) & \text{if } \xi = \omega,\ \omega \neq 0, \pi \\ \rho_4(N) & \text{if } |\xi - \omega| = \frac{2\pi k}{N},\ k = 1, 2, \ldots, N-1 \end{cases}$$

where $|\rho_3(N)| \leq \frac{C}{\sqrt{N}}$ and $|\rho_4(N)| \leq \frac{C}{\sqrt{N}}$.

Frequency - ETFE

Case 1: Periodic Input

When the input $u[n]$ is periodic and N is a multiple of the period, $|U(\omega)|^2$ increases like N for some ω and is zero for others. The number of frequencies $\omega = 2\pi k/N$ for which $|U(\omega)|^2$ is nonzero, and hence for which the ETFE is defined, is fixed and no more than the period length of the signal. Then,

- The ETFE $\widehat{\widehat{G}}(e^{j\omega})$ is defined only for a fixed number of frequencies
- At these frequencies the ETFE is unbiased and its variance decays like $1/N$

NOTE: When $u[n]$ is not periodic, the variance does not decay with N but remains at the noise-to-signal ratio.

Frequency - ETFE

Case 1: Stochastic Input

Lemma 2 shows that the periodogram $|U(\omega)|^2$ is an erratic function of ω, which fluctuates around $\Phi_{uu}(\omega)$, which we assume to be bounded. Lemma 1 thus tells us that

- The ETFE is an asymptotically unbiased estimate of the transfer function at increasingly (with N) many frequencies
- The variance of the ETFE does not decrease as N increases, and it is given as the noise-to-signal ratio at frequency in question
- The estimates at different frequencies are asymptotically uncorrelated

Frequency - Spectral Analysis

In addition to the ETFE $\hat{\hat{G}}(e^{j\omega})$, the relationship (Equation (87) - Lecture 2)

$$\Phi_{uy}(e^{j\omega}) = G(e^{j\omega})\Phi_{uu}(e^{j\omega}) \tag{42}$$

suggests that the frequency response can also be estimated as:

$$\widehat{G}(e^{j\omega}) = \frac{\widehat{\Phi}_{uy}(e^{j\omega})}{\widehat{\Phi}_{uu}(e^{j\omega})}, \tag{43}$$

which reduces to (37) when we take into account (Equations (98)-(99) - Lecture 2)

$$\widehat{\Phi}_{uu}(\omega) = \frac{1}{N}|U(\omega)|^2, \quad \widehat{\Phi}_{yu}(\omega) = \frac{1}{N}Y(\omega)U^*(\omega). \tag{44}$$

Frequency - Spectral Analysis

It is possible to "smooth" the ETFE by redefining the power spectra as

$$\widehat{\Phi}_{uu}(e^{j\omega}) = \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)|U(\xi)|^2 d\xi \quad (45)$$

$$\widehat{\Phi}_{yu}(e^{j\omega}) = \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)Y(\xi)U^*(\xi) d\xi \quad (46)$$

where W_γ, called frequency weighting function or frequency window, satisfies

$$\int_{-\pi}^{\pi} W_\gamma(\xi)d\xi = 1, \quad \int_{-\pi}^{\pi} \xi W_\gamma(\xi)d\xi = 1$$

The width of this window controls the trade-off between bias and variance. A wider window implies more smoothing, less variance and more bias. Typical frequency windows are shown in the figure below.

Frequency - Spectral Analysis

- Solid line: Parzen.
- Dashed line: Hamming.
- Dotted line: Bartlett.