System Identification and Robust Control

Lecture 2: Linear Time Invariant Systems

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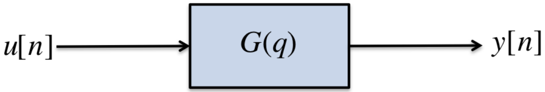
Discrete-time LTI Systems



- A system is said to be *time invariant* if its response to a certain input signal does NOT depend on time.

- A system is said to be *linear* if its output response to a linear combinations of inputs is the same linear combination of the output responses of the individual inputs.

- A system is said to be *causal/strictly causal* if the output at a present time depends on the input up to the present/previous time only.

Discrete-time LTI Systems

For a linear time-invariant (LTI) system with impulse response $g[n]$, the output sequence $y[n]$ is related to the input sequence $u[n]$ through the convolution sum,

$$y[n] = g[n] * u[n] = \sum_{k=-\infty}^{\infty} g[k]u[n-k], \tag{1}$$

where n is an integer number. For causal systems,

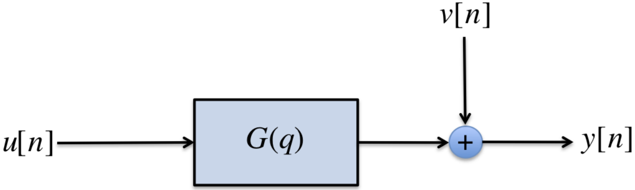
$$y[n] = g[n] * u[n] = \sum_{k=0}^{\infty} g[k]u[n-k], \quad g[k] = 0, k < 0. \tag{2}$$

For strictly causal systems,

$$y[n] = g[n] * u[n] = \sum_{k=1}^{\infty} g[k]u[n-k], \quad g[k] = 0, k \leq 0. \tag{3}$$

Discrete-time LTI Systems

- According to (3), the output can be exactly calculated once the input is known.
- This is unrealistic. There are always uncontrolled signals affecting the system.
- We assume that such effects can be lumped into an additive term $v[n]$ at the output.



$$y[n] = g[n] * u[n] + v[n] = \sum_{k=1}^{\infty} g[k]u[n-k] + v[n]. \tag{4}$$

Discrete-time LTI Systems

- Let $v[n]$ be given as

$$v[n] = h[n] * e[n] = \sum_{k=0}^{\infty} h[k]e[n-k] \tag{5}$$

where $e[n]$ is *white noise*, i.e., a sequence of independent (identically distributed) random variables with a certain probability density function.

- For normalization reasons, we usually assume $h[0] = 1$, which is no loss of generality since the variance of $e[n]$ can be adjusted.

- Note that $e[n]$ and $v[n]$ are *stochastic processes* (i.e., sequences of random variables). The disturbance that we observe and that are added to the system output are thus *realizations* of the *stochastic process* $v[n]$. More on *stochastic processes* later.

Discrete-time LTI Systems

- It is usually convenient to introduce a shorthand notation for sums like (3) and (5). We introduce the *forward shift operator* q by

$$qu[n] = u[n+1]$$

and the *backward shift operator* q^{-1} by

$$q^{-1}u[n] = u[n-1].$$

- We can then rewrite (3) as

$$\sum_{k=1}^{\infty} g[k]u[n-k] = \sum_{k=1}^{\infty} g[k](q^{-k}u[n]) = \left[\sum_{k=1}^{\infty} g[k]q^{-k}\right] u[n] = G(q)u[n]$$

where

$$G(q) = \sum_{k=1}^{\infty} g[k]q^{-k}. \tag{6}$$

Discrete-time LTI Systems

- Similarly, with

$$H(q) = \sum_{k=0}^{\infty} h[k]q^{-k}, \qquad (7)$$

we can write

$$v[n] = H(q)e[n].$$

- The basic description for a linear system with additive disturbances will thus be

$$y[n] = G(q)u[n] + H(q)e[n]. \qquad (8)$$

Discrete-time LTI Systems

- The system G is *stable* if

$$G(q) = \sum_{k=1}^{\infty} g[k]q^{-k}, \;\; \sum_{k=1}^{\infty} |g[k]| < \infty \tag{9}$$

- This definition is consistent with the bounded-input, bounded-output (BIBO) stability. If the input $u[n]$ satisfies $u[n] \leq C$, then the corresponding output $y[n] = G(q)u[n]$ will also be bounded, i.e., $y[n] \leq C'$.

- The system G is *strictly stable* if

$$G(q) = \sum_{k=1}^{\infty} g[k]q^{-k}, \;\; \sum_{k=1}^{\infty} k|g[k]| < \infty \tag{10}$$

- For a rational $G(q)$, stability implies strictly stability and vice versa.

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Frequency Response of LTI Systems

- Consider as an input sequence a complex exponential of radian frequency ω, i.e. $u[n] = e^{j\omega n}$ for $-\infty < n < \infty$. The output of the system is given by

$$y[n] = g[n] * u[n] = \sum_{k=-\infty}^{\infty} g[k]u[n-k] \quad = \quad \sum_{k=-\infty}^{\infty} g[k]e^{j\omega(n-k)} \tag{11}$$

$$= \left(\sum_{k=-\infty}^{\infty} g[k]e^{-j\omega k} \right) e^{j\omega n}.$$

- Defining,

$$G(e^{j\omega}) = \sum_{k=-\infty}^{\infty} g[k]e^{-j\omega k}, \tag{12}$$

we can write the output sequence as

$$y[n] = G(e^{j\omega})e^{j\omega n}. \tag{13}$$

Frequency Response of LTI Systems

- As result we have that the complex exponential $e^{j\omega n}$ is an eigenfunction of the LTI system with associated eigenvalue equal to $G(e^{j\omega})$. The eigenvalue $G(e^{j\omega})$ is called the *Frequency Response* of the system and describes the changes in amplitude and phase of the complex exponential input.

- Note that if

$$u[n] = \cos(\omega n) = Re\{e^{j\omega n}\} \tag{14}$$

we can write the output sequence as

$$y[n] = Re\{G(e^{j\omega})e^{j\omega n}\} = |G(e^{j\omega})| \cos(\omega t + \phi), \tag{15}$$

where $\phi = \arg G(e^{j\omega})$.

Frequency Response of LTI Systems

- An important distinction exists between continuous-time and discrete-time LTI systems. While in the continuous-time domain we need specify the frequency response $G(\Omega)$ over the interval $-\infty < \Omega < \infty$, in the discrete-time domain we only need specify the frequency response $G(e^{j\omega})$ over an interval of length 2π, e.g., $-\pi < \omega \leq \pi$.

- This property is based on the periodicity of the complex exponential. Using the fact that $e^{\pm j2\pi r} = 1$ for any integer r, we can show that

$$e^{-j(\omega+2\pi r)n} = e^{-j\omega n}e^{-j2\pi rn} = e^{-j\omega n}. \tag{16}$$

- As we will show later, a broad class of input signals can be represented by a linear combination of complex exponentials. In this case, the knowledge of the frequency response allows us to find the output of the LTI system.

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Discrete-Time Fourier Transform

- Many sequences can be represented by a Fourier integral of the form

$$u[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(e^{j\omega}) e^{j\omega n} d\omega, \tag{17}$$

where

$$U(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n] e^{-j\omega n}. \tag{18}$$

- The *Inverse Fourier Transform* (17) represents $u[n]$ as a superposition of infinitesimal complex exponentials over the interval $-\pi < \omega \leq \pi$.
- The *Discrete-Time Fourier Transform* (18), or simply *Fourier Transform*, determines how much of each frequency component over the interval $-\pi < \omega \leq \pi$ is required to synthesize $u[n]$ using (17).
- The Fourier Transform is usually referred to as *Spectrum*. Comparing (12) and (18), it is possible to note that the frequency response of a LTI system is the Fourier Transform of the impulse response $h[n]$.
- As we stated above, the frequency response is periodic. Likewise, the Fourier Transform is periodic with period 2π.

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Z-Transform

- Given a sequence $u[n]$, we define the *Z-Transform* as

$$U(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n}. \tag{19}$$

- "Time-shift" property:

$$Z\{u[n-1]\} = z^{-1} U(z), \quad Z\{u[n+1]\} = z U(z).$$

- We can now formally define the *transfer function* of an LTI system $\{g[n]_1^{\infty}\}$ as

$$G(z) = \sum_{n=1}^{\infty} g[n] z^{-n}. \tag{20}$$

Z-Transform

- By comparing (6) and (20) we can understand why we usually informally call $G(q)$ *transfer operator* or *transfer function* of the LTI system.

- Comparing (18) and (19) we note that we can obtain the Fourier Transform evaluating the Z-Transform at the unit circle $(z = e^{j\omega})$.

- Based on this property, the frequency response $G(e^{j\omega})$ of a discrete-time LTI system $g[n]$ can be obtained evaluating the Z-Transform $G(z)$ at $z = e^{j\omega}$.

- **Remark:** Recall that the frequency response $G(j\omega)$ of a continuous-time LTI system $g(t)$ can be obtained evaluating the Laplace Transform $G(s)$ at $s = j\omega$.

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Sampled Signals

- A sequence $u[n]$ is generally a representation of a sampled signal. Given a continuous signal $u(t)$, its sampled version $u_s(t)$ can be written as $u_s(t) = u(t)s(t)$ with
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \tag{21}$$
where δ is the Dirac delta function and T_s is the sampling period.

- In this case we write
$$u_s(t) = u(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} u(nT_s)\delta(t - nT_s), \tag{22}$$
and
$$u[n] = u(nT_s). \tag{23}$$

Sampled Signals

Based on the definition of the *Continuous-Time Fourier Transform*,

$$u[t] = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\Omega)e^{j\Omega t}d\Omega \quad (24)$$

$$U(\Omega) = \int_{-\infty}^{\infty} u(t)e^{-j\Omega t}dt, \quad (25)$$

we can obtain the continuous-time spectrum for the sampled signal

$$U_s(\Omega) = \sum_{n=-\infty}^{\infty} u(nT_s)\int_{-\infty}^{\infty}\delta(t-nT_s)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} u(nT_s)e^{-j\Omega nT_s}. \quad (26)$$

- Using the Discrete-Time Fourier Transform (18) we can compute

$$U(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n]e^{-j\omega n}. \tag{27}$$

- Comparing (26) and (27), and taking into account (23) we conclude that

$$U_s(\Omega) = U(e^{j\omega})\Big|_{\omega=\Omega T_s} = U(e^{j\Omega T_s}). \tag{28}$$

- The Discrete-Time Fourier Transform $U(e^{j\omega})$ is simply a frequency-scaled version of the Continuous-Time Fourier Transform $U_s(\Omega)$ where the scale factor is given by

$$\omega = \Omega T_s = \frac{\Omega}{f_s} = 2\pi\frac{f}{f_s}. \tag{29}$$

Sampled Signals

- Nyquist theorem relates the sampling frequency $f_s = 1/Ts$ with the maximum frequency f_{max} of the signal before sampling.
- In order to avoid aliasing distortion, it is required that

$$f_s > 2f_{max}. \tag{30}$$

- Therefore, every time we sample with frequency f_s we are assuming that the maximum frequency of the signal to be sampled is less than $f_s/2$.
- In other words, we are assuming that

$$U(\Omega) = \begin{cases} \neq 0 & -2\pi\frac{f_s}{2} < \Omega \leq 2\pi\frac{f_s}{2} \\ = 0 & \text{otherwise.} \end{cases} \tag{31}$$

- Based on the scaling (29), we will have

$$U(e^{j\omega}) = \begin{cases} \neq 0 & -\pi < \omega \leq \pi \\ = 0 & \text{otherwise.} \end{cases} \tag{32}$$

  implying that the interval $-\pi < \omega \leq \pi$ in the discrete-time domain corresponds to the interval $-\pi f_s < \Omega \leq \pi f_s$ $(-f_s/2 < f \leq f_s/2)$ in the continuous-time domain.

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Aliasing

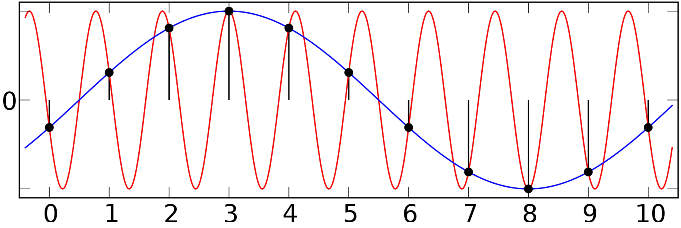

Figure: Time-domain Interpretation

- The original (red) sinusoidal has $f_r \approx 0.9Hz$ and $T_r = 1/0.9 = 1.11s$.
- The signal is sampled every $1s$, therefore with a frequency $f_s = 1Hz$.
- Note that the original (red) sinusoidal will be seen as a ficticious (blue) sinusoidal of $f_b \approx 0.1Hz$ and $T_b = 10s$!!!!

http://www.dsptutor.freeuk.com/aliasing/AD102.html

Aliasing

- We can write the periodic function $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, with period T_s, by a Fourier series.

- Then,

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_s}t} \tag{33}$$

where

$$\begin{aligned} c_k &= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \sum_{n=-\infty}^{\infty} \delta(\tau - nT_s) e^{-j\frac{2\pi k}{T_s}\tau} d\tau \\ &= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(\tau) e^{-j\frac{2\pi k}{T_s}\tau} d\tau \\ &= \frac{1}{T_s} \end{aligned}$$

Aliasing

- By defining $\omega_s = 2\pi f_s = \frac{2\pi}{T_s}$, we can then write

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \tag{34}$$

- Based on the definition of the *Continuous-Time Fourier Transform* (25), we can obtain the continuous-time spectrum for the sampled signal $u_s(t) = u(t)s(t)$ as

$$\begin{aligned} U_s(\Omega) &= \int_{-\infty}^{\infty} u(\tau) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s \tau} e^{-j\Omega\tau} d\tau \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau) e^{-j(\Omega - k\omega_s)\tau} d\tau \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} U(\Omega - k\omega_s) \end{aligned} \tag{35}$$

where $U(\Omega)$ is the continuous-time spectrum of $u(t)$.

Aliasing

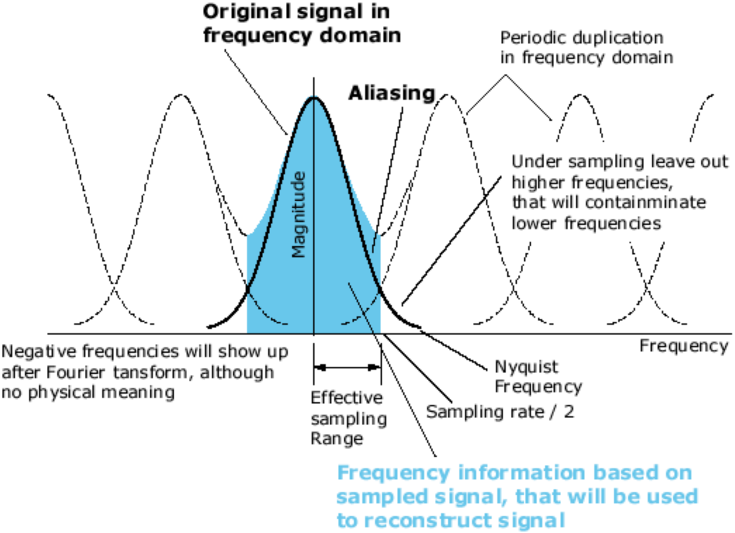

Figure: Frequency-domain Interpretation

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Discrete Fourier Series

- We come back now to the idea of representing signals by a linear combination of complex exponential and we consider at this time the periodic sequence $\tilde{u}[n]$ with period N, i.e. $\tilde{u}[n] = \tilde{u}[n + rN]$ for any integer r.

- As in the continuous case, we can represent $\tilde{u}[n]$ by its *Fourier Series*,

$$\tilde{u}[n] = \frac{1}{N} \sum_{k} \tilde{U}[k] e^{j\frac{2\pi}{N}kn}. \tag{36}$$

- By the *Fourier Series*, the periodic sequence is represented as a sum of complex exponentials with frequencies that are integer multiples of the fundamental frequency $2\pi/N$.

- We say that these are harmonically related complex exponentials.

Discrete Fourier Series

- The *Fourier Series* representing continuous-time periodic signals require an infinite number of harmonically related complex exponentials, whereas the *Fourier Series* for any discrete-time periodic signal requires only N harmonically related complex exponentials.
- This is explained by the periodicity of the complex exponential (16).
- Thus, the *Discrete Fourier Series* of the periodic sequence $\tilde{u}[n]$ with period N can be written as

$$\tilde{u}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{U}[k] e^{j\frac{2\pi}{N}kn}, \tag{37}$$

  where the *Fourier Series* coefficients have the form

$$\tilde{U}[k] = \sum_{n=0}^{N-1} \tilde{u}[n] e^{-j\frac{2\pi}{N}kn}. \tag{38}$$

- The sequence $\tilde{U}[k]$ is periodic with period N.

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Representing Periodic Sequences

- Sequences that can be expressed as a sum of complex exponentials, as it is the case for all periodic sequences, can be considered to have a Fourier Transform as a train of impulses.

- It is simple to demonstrate that the expression

$$U(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi r), \tag{39}$$

  where we assume that $-\pi < \omega_o \leq \pi$, corresponds to the Fourier Transform of the complex exponential sequence $e^{j\omega_o n}$.

- To show that, we replace the expression in (17) to obtain

$$u[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\omega})e^{j\omega n}d\omega = \int_{-\pi}^{\pi} \delta(\omega - \omega_o)e^{j\omega n}d\omega = e^{j\omega_o n}.$$

Representing Periodic Sequences

- Then, if a sequence $u[n]$ can be represented as a sum of complex exponentials, i. e.,

$$u[n] = \sum_k a_k e^{j\omega_k n} \quad (40)$$

for $\infty < n < \infty$, it has a Fourier Transform given by

$$U(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k a_k 2\pi\delta(\omega - \omega_k + 2\pi r). \quad (41)$$

- This means that if $\tilde{u}[n]$ is periodic with period N and Discrete Fourier Series coefficients $\tilde{U}[k]$, we can write

$$\tilde{u}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{U}[k] e^{j\frac{2\pi}{N}kn}, \quad (42)$$

and the Fourier Transform $\tilde{U}(e^{j\omega})$ is defined to be the impulse train

$$\tilde{U}(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} 2\pi \frac{\tilde{U}[k]}{N} \delta(\omega - \frac{2\pi k}{N} + 2\pi r) = \sum_{k=-\infty}^{\infty} 2\pi \frac{\tilde{U}[k]}{N} \delta(\omega - \frac{2\pi k}{N}). \quad (43)$$

Representing Periodic Sequences

- As an example, consider now a periodic impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1 & n = rN \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

  where r is an integer and N is the period.

- We can compute first the Discrete Fourier Series coefficients

$$\tilde{P}[k] = \sum_{n=0}^{N-1} \tilde{p}[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \delta[n] e^{-j\frac{2\pi}{N}kn} = 1. \tag{45}$$

- Therefore, according to (43) the Fourier Transform is given by

$$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta(\omega - \frac{2\pi k}{N}). \tag{46}$$

- The Fourier Transform of the periodic impulse train becomes important when we want to relate finite-length and periodic sequences.

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Finite-length $\rightarrow$ Periodic Sequences

- Consider now a finite length sequence $u[n]$ ($u[n] = 0$ everywhere except over the interval $0 \leq n \leq N-1$).
- We can construct an associated periodic sequence $\tilde{u}[n]$ as the convolution of the finite-length sequence with the impulse train (44) of period N:

$$\tilde{u}[n] = u[n] * \tilde{p}[n] = u[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} u[n - rN].$$

- The periodic sequence $\tilde{u}[n]$ is a set of periodically repeated copies of the finite-length sequence $u[n]$.
- Assuming that the Fourier Transform of $u[n]$ is $U(e^{j\omega})$, and recalling that the Fourier Transform of a convolution is the product of the Fourier Transforms, we can obtain the Fourier Transform for $\tilde{u}[n]$ as

$$\tilde{U}(e^{j\omega}) = U(e^{j\omega})\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} U(e^{j\frac{2\pi k}{N}})\delta(\omega - \frac{2\pi k}{N}),$$

where we have used (46).

Finite-length → Periodic Sequences

- This result must be coincident with our definition (43) and therefore it must be
$$\tilde{U}[k] = U(e^{j\frac{2\pi k}{N}}) = U(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} \ . \qquad (47)$$

- This very important result implies that the periodic sequence $\tilde{U}[k]$ with period N of Discrete Fourier Series coefficients are equally spaced samples of the Fourier Transform of the finite-length sequence $u[n]$ obtained by extracting one period of $\tilde{u}[n]$.

- This corresponds to sample the Fourier Transform at N equally spaced frequencies over the interval $-\pi < \omega \leq \pi$ with spacing $2\pi/N$.

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Discrete Fourier Transform

- As we defined

$$u[n] = \begin{cases} \tilde{u}[n] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \qquad (48)$$

$$\tilde{u}[n] = u[(n \text{ modulo } N)] \qquad (49)$$

- We define now for consistency (and to maintain duality between time and frequency),

$$U[k] = \begin{cases} \tilde{U}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases} \qquad (50)$$

$$\tilde{U}[k] = U[(k \text{ modulo } N)]. \qquad (51)$$

- We have used the fact that the Discrete Fourier Series sequence $\tilde{U}[k]$ is itself a sequence with period N.

Discrete Fourier Transform

- The sequence $U[k]$ is named *Discrete Fourier Transform* and is written as

$$U[k] = \sum_{n=0}^{N-1} u[n] e^{-j\frac{2\pi}{N}kn}, \quad (52)$$

$$u[n] = \frac{1}{N} \sum_{k=0}^{N-1} U[k] e^{j\frac{2\pi}{N}kn}. \quad (53)$$

- The *Discrete Fourier Transform* (53) gives us the *Discrete-Time Fourier Transform* (or simply the *Fourier Transform*) (18) at N equally spaced frequencies over the interval $0 \leq \omega \leq 2\pi$ (or $-\pi < \omega \leq \pi$):

$$U[k] = U(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}, \quad 0 \leq k \leq N-1. \quad (54)$$

Discrete Fourier Transform

- In addition to its theoretical importance as a Fourier representation of sequences, the Discrete Fourier Transform (DFT) plays a central role in digital signal processing because there exist efficient algorithms for its computation.
- These algorithms are usually referred as Fast Fourier Transform (FFT). The FFT is simply an efficient implementation of the DFT.
- In applications based on Fourier analysis of signals, it is the Discrete-Time Fourier Transform (FT) that is desired, while it is the Discrete Fourier Transform (DFT) that is actually computed.
- For finite-length signals, the DFT provides frequency-domain samples of the FT and the implications of this sample must be clearly understood and accounted for.

Discrete Fourier Transform

- Given a sequence $u[n]$ of N points sampled at frequency f_s, we can compute the DFT via the FFT as

$$U[k] = \sum_{n=0}^{N-1} u[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \le k \le N-1 \tag{55}$$

$$u[n] = \frac{1}{N} \sum_{k=0}^{N-1} U[k] e^{j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1. \tag{56}$$

- Considering the relationship (54) between DFT and FT, we can write the spectrum as

$$U(\omega) = U(e^{j\omega}) = \sum_{n=0}^{N-1} u[n] e^{-j\omega n}, \omega = \frac{2\pi}{N} k (0 \le k \le N-1; 0 \le \omega < 2\pi). \tag{57}$$

Discrete Fourier Transform

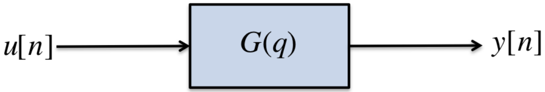
- Considering the scaling (29) between sequences and sampled signals, we can write the spectrum as

$$U(\Omega) = \sum_{n=0}^{N-1} u[n]e^{-j\Omega T_s n}, \Omega = \frac{2\pi k}{NT_s}(0 \leq k \leq N-1; 0 \leq \Omega < \frac{2\pi}{T_s} = 2\Omega_{max})$$

$$U(f) = \sum_{n=0}^{N-1} u[n]e^{-j\frac{2\pi f}{f_s}n}, f = \frac{kf_s}{N}(0 \leq k \leq N-1; 0 \leq f < f_s = 2f_{max})$$

- In addition to the periodicity of the DFT ($U(\omega + 2\pi) = U(\omega)$), we have that $U(-\omega) = \overline{U(\omega)}$ for real $u[n]$. Therefore, the function $U(\omega)$ is uniquely defined by its values over the interval $[0, \pi]$.
- We associate high frequencies with frequencies close to π and low frequencies with frequencies close to 0.
- As a consequence of these properties, it is exactly the same to define $U(\omega)$ over the interval $0 \leq \omega < 2\pi$ or the interval $-\pi < \omega \leq \pi$. The DFT will give the values of the FT over any of these intervals with a frequency spacing equal to $2\pi/N$.

Discrete Fourier Transform - Filtering



Theorem: Let $y[n]$ and $u[n]$ be related by strictly stable system $G(q)$:

$$y[n] = G(q)u[n].$$

The input $u[n]$ for $n \leq 0$ is unknown but obeys $|u[n]| \leq C_u$ for all n. Let $U(\omega)$ and $Y(\omega)$ denote the DFT of $u[n]$ and $y[n]$ respectively at $\omega = \frac{2\pi}{N}k$ with $0 \leq k \leq N-1$, $0 \leq \omega < 2\pi$ defined as in (57). Then,

$$Y(\omega) = G(e^{i\omega})U(\omega) + R(\omega) \tag{58}$$

where $|R(\omega)| \leq 2C_uC_G \quad C_G = \sum_{k=1}^{\infty} k|g[k]|$.

Corolary: Suppose $u[n]$ is periodic with period N. Then $R(\omega)$ in (58) is zero for $\omega = 2\pi k/N$.

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Stochastic Processes

- Until now, we have assumed that the signals are deterministic. Sometimes, the mechanism of signal generation is so complex that it is very difficult, if not impossible, to represent the signal as deterministic. In these cases, modeling the signal as an outcome of a random variable is extremely useful.

- Each individual sample $u[n]$ of a particular signal is assumed to be an outcome of a random variable $\mathbf{u}_n$. The entire signal is represented by a collection of such random variables, one for each sample time, $-\infty < n < \infty$.

- The collection of these random variables is called a random process. We assume that the sequence $u[n]$ for $-\infty < n < \infty$ is generated by a random process with specific probability distribution that underlies the signal.

Stochastic Processes

- An individual random variable $\mathbf{u}_n$ is described by the probability distribution function

$$F_{\mathbf{u}_n}(u_n, n) = \text{Probability}[\mathbf{u}_n \leq u_n], \tag{59}$$

where $\mathbf{u}_n$ denotes the random variable and u_n is a particular value of $\mathbf{u}_n$.

- If $\mathbf{u}_n$ takes on a continuous range of values, it can be specified by the probability density function

$$f_{\mathbf{u}_n}(u_n, n) = \frac{\partial F_{\mathbf{u}_n}(u_n, n)}{\partial u_n} \tag{60}$$

$$F_{\mathbf{u}_n}(u_n, n) = \int_{-\infty}^{u_n} f_{\mathbf{u}_n}(u, n) du. \tag{61}$$

- When we have two stochastic processes $\mathbf{u}_n$ and $\mathbf{v}_n$, the interdependence is described by the joint probability distribution function

$$F_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) = \text{Probability}[\mathbf{u}_n \leq u_n \text{ and } \mathbf{v}_m \leq v_m], \qquad (62)$$

and by the joint probability density

$$f_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) = \frac{\partial^2 F_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m)}{\partial u_n \partial v_m}. \qquad (63)$$

- When $F_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) = F_{\mathbf{u}_n}(u_n, n)F_{\mathbf{v}_m}(v_m, m)$ we say that the processes are independent.

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Ensemble Averages

- It is often useful to characterize a random variable in terms of its mean, variance and autocorrelation. The mean of a random process $\mathbf{u}_n$ is defined as

$$m_u[n] = m_{\mathbf{u}_n} = E\{\mathbf{u}_n\} = \int_{-\infty}^{\infty} u f_{\mathbf{u}_n}(u,n) du, \tag{64}$$

where E denotes and operator called mathematical expectation. Keeping in mind that the sequence $u[n]$ is the outcome of the random variable $\mathbf{u}_n$, we can simplify the notation writing alternatively the mean of the sequence $u[n]$ as

$$m_u[n] = E\{u[n]\} = \int_{-\infty}^{\infty} u f_{\mathbf{u}_n}(u_n,n) du, \tag{65}$$

- The variance of $\mathbf{u}_n$ is defined as

$$\sigma_u^2[n] = \sigma_{\mathbf{u}_n}^2 = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})^2\} = \int_{-\infty}^{\infty} (u - m_{\mathbf{u}_n})^2 f_{\mathbf{u}_n}(u,n) du. \tag{66}$$

or alternatively

$$\sigma_u^2[n] = E\{(u[n] - m_u[n])^2\} = \int_{-\infty}^{\infty} (u - m_{\mathbf{u}_n})^2 f_{\mathbf{u}_n}(u,n) du. \tag{67}$$

Ensemble Averages

- The autocorrelation of $\mathbf{u}_n$ is defined as

$$R_{uu}[n,m] = E\{\mathbf{u}_n \mathbf{u}_m^*\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_n u_m^* f_{\mathbf{u}_n,\mathbf{u}_m}(u_n, n, u_m, m) du_n du_m, \tag{68}$$

  whereas the autocovariance sequence is defined as

$$C_{uu}[n,m] = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})(\mathbf{u}_m - m_{\mathbf{u}_m})^*\} = R_{uu}[n,m] - m_{\mathbf{u}_n} m_{\mathbf{u}_m}^*. \tag{69}$$

- In the same way, given two stochastic processes $\mathbf{u}_n$ and $\mathbf{v}_n$ we can define the cross-correlation as

$$R_{uv}[n,m] = E\{\mathbf{u}_n \mathbf{v}_m^*\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_n v_m^* f_{\mathbf{u}_n,\mathbf{v}_m}(u_n, n, v_m, m) du_n dv_m, \tag{70}$$

  whereas the cross-covariance sequence is defined as

$$C_{uv}[n,m] = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})(\mathbf{v}_m - m_{\mathbf{v}_m})^*\} = R_{uv}[n,m] - m_{\mathbf{u}_n} m_{\mathbf{v}_m}^*. \tag{71}$$

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Stationary Stochastic Processes

- In general the statistical properties of a random variable may depend on n.
- For a *stationary process* the statistical properties are invariant to a shift of time origin. This means that the first order averages such as the mean and variance are independent of time and the second order averages such as the autocorrelation are dependent on the time diference.
- Thus, for a stationary process we can write

$$m_u[n] = m_u = E\{\mathbf{u}_n\} \quad (72)$$

$$\sigma_u^2[n] = \sigma_u^2 = E\{(\mathbf{u}_n - m_{\mathbf{u}})^2\} \quad (73)$$

$$R_{uu}[n+m, n] = R_{uu}[m] = E\{\mathbf{u}_{n+m}\mathbf{u}_n^*\}. \quad (74)$$

- In many cases, the random processes are not stationary in the *strict sense* because their probability distributions are not time invariant but (72)–(74) still hold. We name those random processes as *wide-sense stationary*.

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Power Spectrum Density

- While stochastic signals are not absolutely summable or square summable and consequently do not have Fourier Transforms, many of the properties of such signals can be summarized in terms of the autocorrelation or autocovariance sequence, for which the Fourier Transform often exists.

- We define the *Power Spectrum Density* (PSD) as the Fourier Transform of the auto-covariance sequence

$$\Phi_{uu}(\omega) = \Phi_{uu}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} C_{uu}[n] e^{-j\omega n}, \tag{75}$$

and the *Cross Spectrum Density* (CSD) as the Fourier Transform of the cross-covariance sequence

$$\Phi_{uv}(\omega) = \Phi_{uv}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} C_{uv}[n] e^{-j\omega n}. \tag{76}$$

Power Spectrum Density

- By definition of the auto-covariance and the Inverse Fourier Transform we can write

$$\sigma_u^2 = E\{(\mathbf{u}_n - m_{\mathbf{u}_n})^2\} = C_{uu}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{uu}(\omega) d\omega. \tag{77}$$

- Assuming that the sequence $u[n]$ is the sampled version of a stationary random signal $s(t)$ whose PSD $\Phi_{ss}(\Omega)$ is bandlimited by the antialiasing lowpass filter ($-2\pi\frac{f_s}{2} < \Omega < 2\pi\frac{f_s}{2}$), its PSD $\Phi_{uu}(\omega)$ is proportional to $\Phi_{ss}(\Omega)$ over the bandwith of the antialiasing filter, i.e.,

$$\Phi_{uu}(\omega) = \frac{1}{T_s}\Phi_{ss}\left(\frac{\omega}{T_s}\right), \quad |\omega| < \pi \tag{78}$$

$$\Phi_{uu}(f) = \Phi_{ss}\left(\frac{\omega}{T_s}\right) = \frac{\Phi_{uu}(\omega)}{f_s}, \quad |\omega| < \pi, f = \frac{\omega f_s}{2\pi}. \tag{79}$$

Power Spectrum Density - Filtering

- For a linear time-invariant (LTI) system with impulse response $g[n]$, we know that the output sequence $y[n]$ is related to the input sequence $u[n]$ through the convolution sum,

$$y[n] = g[n] * u[n] = \sum_{k=-\infty}^{\infty} g[k]u[n-k]. \tag{80}$$

- We assume for convenience that $m_u = 0$. Then we have

$$m_y = E\{y[n]\} = \sum_{k=-\infty}^{\infty} g[k]E\{u[n-k]\} = \sum_{k=-\infty}^{\infty} g[k]m_u = 0. \tag{81}$$

Power Spectrum Density - Filtering

- The autocorrelation sequence for the output $y[n]$ is given by

$$\begin{aligned}
R_{yy}[\tau] &= E\{y[n]y[n-\tau]\} \\
&= E\left\{ \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} g[k]u[n-k]g[r]u[n-\tau-r] \right\} \\
&= \sum_{k=-\infty}^{\infty} g[k] \sum_{r=-\infty}^{\infty} g[r]E\{u[n-k]u[n-\tau-r]\} \\
&= \sum_{k=-\infty}^{\infty} g[k] \sum_{r=-\infty}^{\infty} g[r]R_{uu}[\tau+r-k]
\end{aligned}$$

Power Spectrum Density - Filtering

- By making the substitution $-l = r - k$ we can write

$$
\begin{aligned}
R_{yy}[\tau] &= \sum_{l=-\infty}^{\infty} R_{uu}[\tau - l] \sum_{r=-\infty}^{\infty} g[l+r]g[r] \\
R_{yy}[\tau] &= \sum_{l=-\infty}^{\infty} R_{uu}[\tau - l] R_{gg}[l]. \qquad (82)
\end{aligned}
$$

- Taking into account that the Fourier Tranform of $R_{gg}(l) = g[n] * g[-n]$ is equal to $G(e^{j\omega})G^*(e^{j\omega}) = \left|G(e^{j\omega})\right|^2$ and applying Fourier Transform to the last equation we can obtain the relationship

$$
\Phi_{yy}(e^{j\omega}) = \left|G(e^{j\omega})\right|^2 \Phi_{uu}(e^{j\omega}). \qquad (83)
$$

Power Spectrum Density - Filtering

- The stochastic process described by

$$v[n] = H(q)e[n], \tag{84}$$

where $e[n]$ is a sequence of independent random variables with zero mean values and variances λ, has the PSD

$$\Phi_{vv}(e^{j\omega}) = \lambda \left| H(e^{j\omega}) \right|^2 . \tag{85}$$

- **Spectral Factorization:** Suppose that $\Phi_{vv}(e^{j\omega}) > 0$ is a rational function of $e^{j\omega}$. Then, there always exists a monic rational function of z, $H(z)$, with no poles and no zeros on or outside the unit circle such that (85) is satisfied. The spectral factorization concept is important since it provides a way of representing the disturbance in the standard form $v = H(q)e$ from information about its PSD only.

Power Spectrum Density - Filtering

- The cross-correlation between the input $u[n]$ and output $y[n]$ is given by

$$\begin{aligned} R_{uy}[\tau] &= E\{u[n]y[n-\tau]\} \\ &= E\left\{u[n]\sum_{k=-\infty}^{\infty} g[k]u[n-\tau-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} g[k]E\{u[n]u[n-\tau-k]\} \\ &= \sum_{k=-\infty}^{\infty} g[k]R_{uu}[\tau+k] \\ &= g[\tau]*R_{uu}[-\tau] \end{aligned}$$

- Invoking properties of the autocorrelation, $R_{uu}[-\tau] = R_{uu}[\tau]$, we can write

$$R_{uy}[\tau] = g[\tau]*R_{uu}[\tau] \quad (86)$$

- Applying Fourier Transform to the last equation we can obtain

$$\Phi_{uy}(e^{j\omega}) = G(e^{j\omega})\Phi_{uu}(e^{j\omega}). \quad (87)$$

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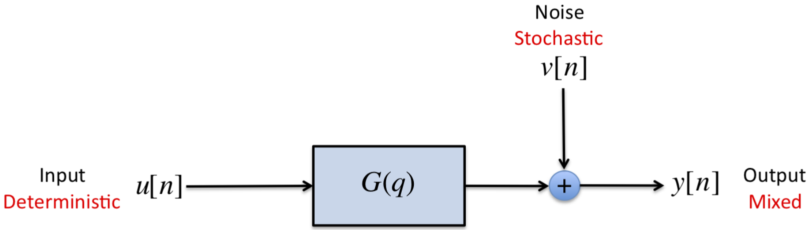
Common Framework

- We consider the input sequence as deterministic and the noise or disturbance as stochastic.
- As a result, the output becomes a stochastic process with deterministic components.



$$Ey[n] = EG(q)u[n] + EH(q)e[n] = G(q)u[n] + H(q)Ee[n] = G(q)u[n].$$

We loose stationarity!!!!

Quasi-stationary Stochastic Processes

- A sequence $s[n]$ is quasi-stationary if it is subject to

$$E\{s[n]\} = m_s[n], \quad |m_s[n]| \leq C \quad \forall n \tag{88}$$

$$E\{s[n]s[m]\} = R_s[n,m], \quad |R_s[n,m]| \leq C \tag{89}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} R_s[n, n-\tau] = R_s(\tau), \quad \forall\tau$$

  - If $s[n]$ is a stationary stochastic process, (88)–(89) are trivially satisfied.
  - if $s[n]$ is a deterministic sequence, the expectation is without effect and quasi-stationarity means that $s[n]$ is bounded, i.e., $|s[n]| < C$, and the following limit exists:

$$R_s(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s[n]s[n-\tau].$$

Quasi-stationary Stochastic Processes

- Let the sequence $y[n]$ be given by

$$y[n] = G(q)u[n] + H(q)e[n], \tag{90}$$

where $u[n]$ is a zero-mean, quasi-stationary, deterministic sequence with spectrum $\Phi_{uu}(e^{j\omega})$ and $e[n]$ is white noise with variance λ.

- Let $G(q)$ and $H(q)$ be stable filters. Then $y[n]$ is quasi-stationary and

$$\begin{aligned} \Phi_{yy}(e^{j\omega}) &= \left|G(e^{j\omega})\right|^2 \Phi_{uu}(e^{j\omega}) + \lambda \left|H(e^{j\omega})\right|^2 & (91) \\ \Phi_{uy}(e^{j\omega}) &= G(e^{j\omega})\Phi_{uu}(e^{j\omega}). & (92) \end{aligned}$$

- This results from the fact that for $s[n] = u[n] + v[n]$ where $u[n]$ is a zero-mean deterministic signal with spectrum $\Phi_{uu}(e^{j\omega})$ and $v[n]$ is a zero-mean stationary stochastic process with spectrum $\Phi_{vv}(e^{j\omega})$, then
  - $(R_{uv}[\tau] = 0)$ $R_{ss}[\tau] = R_{uu}[\tau] + R_{vv}[\tau]$
  - $\Phi_{ss}(e^{j\omega}) = \Phi_{uu}(e^{j\omega}) + \Phi_{vv}(e^{j\omega})$

Relationship DFT-PSD

- Suppose that $s[n]$ is quasi-stationary with power spectrum $\Phi_{ss}(e^{j\omega})$.
- Then, the square of the DFT converges weakly to the power spectrum,

$$E\{|S(\omega)|^2\} \longrightarrow \Phi_{ss}(e^{j\omega}).$$

- Thus,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} E\{|S(\omega)|^2\}\Psi(\omega)d\omega = \int_{-\pi}^{\pi} \Phi_{ss}(\omega)\Psi(\omega)d\omega$$

with $\Psi(\omega)$ a sufficiently smooth function for $|\omega| \leq \pi$ with Fourier coefficients a_τ such that $\sum_{\tau=-\infty}^{\infty} |a_\tau|$. Recall that

$$S(\omega) = S(e^{j\omega}) = \sum_{n=0}^{N-1} s[n]e^{-j\omega n},$$

where $\omega = \frac{2\pi}{N}k\,(0 \leq k \leq N-1; 0 \leq \omega < 2\pi)$.

Relationship DFT-PSD

- Suppose that $s[n]$ is a zero-mean stationary stochastic process with auto-correlation $R_{ss}[\tau]$ and power spectrum $\Phi_{ss}(e^{j\omega})$.

- Then, the square of the DFT converges to the power spectrum,

$$E\{|S(\omega)|^2\} \longrightarrow \Phi_{ss}(e^{j\omega})$$

as $N \to \infty$ assuming that $\sum_{\tau=-\infty}^{\infty} |\tau R_{ss}[\tau]| < \infty$.

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Sample Averages

- For a stationary random process, the essential characteristics of the process are represented by averages such as the mean, variance or autocorrelation.
- Therefore, it is essential to be able to estimate these quantities from finite-length segments of data.
- An estimator for the mean value is the *sample mean*, defined as

$$\widehat{m}_u = \frac{1}{N} \sum_{n=0}^{N-1} u[n], \tag{93}$$

  which is unbiased ($E\{\widehat{m}_u\} = m_u$).
- An estimator for the variance is the *sample variance*, defined as

$$\widehat{\sigma}_u^2 = \frac{1}{N} \sum_{n=0}^{N-1} (u[n] - \widehat{m}_u)^2, \tag{94}$$

  which is asymptotically unbiased ($\lim_{N\to\infty} E\{\widehat{\sigma}_u^2\} = \sigma_u^2$).

Sample Averages

- The estimators for the auto-covariance and cross-covariance are respectively defined as

$$\widehat{C}_{uu}[\tau] = \frac{1}{N}\sum_{n=0}^{N-1}(u[n]-\widehat{m}_u)(u[n-\tau]-\widehat{m}_u), \quad (95)$$

$$\widehat{C}_{uv}[\tau] = \frac{1}{N}\sum_{n=0}^{N-1}(u[n]-\widehat{m}_u)(v[n-\tau]-\widehat{m}_v). \quad (96)$$

- Both estimators are asymptotically unbiased
  - $\lim_{N\to\infty} E\{\widehat{C}_{uu}[\tau]\} = C_{uu}[\tau]$
  - $\lim_{N\to\infty} E\{\widehat{C}_{uv}[\tau]\} = C_{uv}[\tau]$
- In addition it can be showed that

$$E\{\widehat{C}[\tau]\} = \frac{N-|\tau|}{N}C[\tau]. \quad (97)$$

Sample Averages

- It is possible to show that

$$\widehat{\Phi}_{uu}(\omega) = \sum_{n=-\infty}^{\infty} \widehat{C}_{uu}[n]e^{-j\omega n} = \frac{1}{N}|U(\omega)|^2 = P_{uu}(\omega), \qquad (98)$$

and

$$\widehat{\Phi}_{uv}(\omega) = \sum_{n=-\infty}^{\infty} \widehat{C}_{uv}[n]e^{-j\omega n} = \frac{1}{N}U(\omega)V^*(\omega), \qquad (99)$$

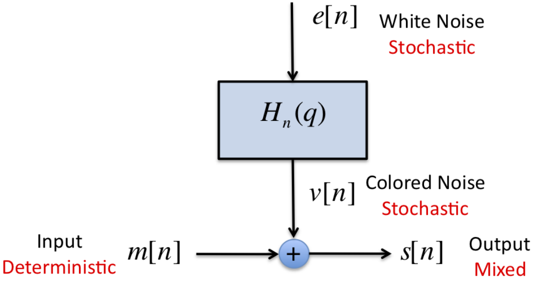
where $U(\omega)$ and $V(\omega)$ are the Discrete Fourier Transforms of $u[n]$ and $v[n]$ respectively and $P_{uu}(\omega)$ is the *Periodogram* of the sequence $u[n]$.

Ergodicity

- Let $s[n]$ be a quasi-stationary sequence. Let $E\{s[n]\} = m[n]$.
- We assume that

$$s[n] - m[n] = v[n] = \sum_{k=0}^{\infty} h_n[k]e[n-k] = H_n(q)e[n] \tag{100}$$

where $e[n]$ is a sequence of independent random variables with zero mean values and $E\{e^2[n]\} = \lambda_n$. $H_n(q)$ for $n = 1, 2, \ldots$ is a uniformly stable family of filters.



Ergodicity

- Then with probability 1 as N tends to infinity,

$$\widehat{R}_{ss}[\tau] = \frac{1}{N}\sum_{n=0}^{N-1} s[n]s[n-\tau] \rightarrow R_{ss}[\tau] \tag{101}$$

$$\widehat{R}_{sm}[\tau] = \frac{1}{N}\sum_{n=0}^{N-1} s[n]m[n-\tau] \rightarrow R_{sm}[\tau] \tag{102}$$

$$\widehat{R}_{sv}[\tau] = \frac{1}{N}\sum_{n=0}^{N-1} s[n]v[n-\tau] \rightarrow R_{sv}[\tau] \tag{103}$$

- This also implies that $\widehat{\Phi}_{ss}(\omega) \rightarrow \Phi_{ss}(\omega)$, $\widehat{\Phi}_{sm}(\omega) \rightarrow \Phi_{sm}(\omega)$ and $\widehat{\Phi}_{sv}(\omega) \rightarrow \Phi_{sv}(\omega)$.

Ergodicity

- This result is quite important. It says that, provided the stochastic part of a sequence can be described as a filtered white noise, the correlation and power spectrum of an observed single realization of $s[n]$, computed as for a deterministic signal (sample averages), coincides, with probability 1 as $N \to \infty$, with that of the stochastic process $s[n]$ (ensemble averages) computed through the expectation operator E.

- Most computations in system identification depend on given realizations of a quasi-stationary process.

- Ergodicity allows us to make statements about repeated experiments.