Nonlinear Systems and Control

Lecture 8 (Meetings 26-28)
Chapter 5: Input-Output Stability
Chapter 6: Passivity
Chapter 14: Passivity-Based Control

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$\mathcal{L}$ Stability

Broader concept that Input-State Stability (ISS).

We consider a system with input-output relation

$$y = Hu, \qquad u : [0, \infty) \to R^m$$

Space of piecewise continuous, bounded functions:

$$\mathcal{L}_\infty^m : \qquad \|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$$

Space of piecewise continuous, square-integrable functions:

$$\mathcal{L}_2^m : \qquad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)dt} < \infty$$

More generally, the space $\mathcal{L}_p^m$ for $1 \leq p < \infty$ (p: p-norm, m : dimension of u)

$$\|u\|_{\mathcal{L}_p^m} = \left(\int_0^\infty \|u(t)\|^p dt\right)^{1/p} < \infty$$

Input-Output Stability

Truncated norm for $1 \leq p \leq \infty$:

$$\|u_\tau\|_{\mathcal{L}_p^m} = \begin{cases} \left(\int_0^\tau \|u(t)\|^p dt\right)^{1/p} & p < \infty \\ \sup_{t\in[0,\tau]} \|u(t)\| & p = \infty \end{cases}$$

allows for the definition of the extended space $\mathcal{L}_{p,e}^m$, i.e.,

$$\mathcal{L}_{p,e}^m = \{u | u_\tau \in \mathcal{L}_p^m, \forall \tau \in [0,\infty)\}, \quad u_\tau(t) = \begin{cases} u(t) & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$$

Example: Exponential functions are in this space.

Input-Output Stability

Note: We use $\mathcal{L}$ to denote any $\mathcal{L}_p$ norm/space without committing to any particular p.

Definition 5.1: A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is *$\mathcal{L}$ stable* if there is a class $\mathcal{K}$ function α and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \alpha\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. It is *finite-gain $\mathcal{L}$ stable* if there are nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma\|u_\tau\|_{\mathcal{L}} + \beta$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. The constant β is called the bias term and is included in the definition to allow for systems where Hu does not vanish at $u = 0$.

Let us consider a linear system. Let $y = Hu$ be described by the convolution operator

$$y(t) = \int_0^t h(t-\sigma)u(\sigma)d\sigma, \qquad h(t) \equiv 0 \text{ for } t < 0$$

Theorem (Young's Convolution Theorem): (See Example 5.2)

$$\|y_\tau\|_{\mathcal{L}_p} = \|(h * u)_\tau\|_{\mathcal{L}_p} \leq \|h\|_{\mathcal{L}_1}\|u_\tau\|_{\mathcal{L}_p}, \qquad \forall p \in [1,\infty], \quad \forall \tau \in [0,\infty)$$

Linear systems are finite-gain $\mathcal{L}_p$ stable if $h \in \mathcal{L}_1$, i.e.

$$\|h\|_{\mathcal{L}_1} = \int_0^\infty |h(\sigma)|d\sigma < \infty$$

In this case, the gain γ is equal to the $\mathcal{L}_1$ norm of the impulse response ($\|h\|_{\mathcal{L}_1} < \infty$).

Holder's Inequality: If $p, q \in [1,\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for every $\tau \in [0,\infty)$

$$\int_0^\tau |f(t)g(t)|dt \leq \left(\int_0^\tau |f(t)|^p dt\right)^{1/p} \left(\int_0^\tau |g(t)|^q dt\right)^{1/q}, \quad \forall t \geq 0$$

$\mathcal{L}_2$ Stability

Theorem 5.4: Consider the linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$. Then, the $\mathcal{L}_2$ gain of the system is $\|G(j\omega)\|_{\mathcal{H}_\infty} = \sup_{\omega \in R} \|G(j\omega)\|_2$, i.e. $\|y\|_{\mathcal{L}_2} \leq \|G(j\omega)\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}_2}$.

Fourier Transform:

$$Y(j\omega) = \int_0^\infty y(t) e^{-j\omega t} dt$$

Parseval's Theorem:

$$\int_0^\infty y^T(t) y(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y^*(j\omega) Y(j\omega) d\omega$$

Proof: Easy. Note that $\|y\|_{\mathcal{L}_2}^2 = \int_0^\infty y^T(t) y(t) dt$. Use Parseval theorem and the fact that $Y(j\omega) = G(j\omega) U(j\omega)$. See proof in the book.

Note: This theorem is the foundation of $\mathcal{H}_\infty$ Control $\rightarrow$ Robust Control.

For LTI systems we can find the exact $\mathcal{L}_2$ gain. For general systems, we can only find an upper bound on the $\mathcal{L}_2$ gain, as the next theorem states.

Theorem 5.5: Consider the nonlinear time-invariant system

$$\dot{x} = f(x) + G(x)u, \quad x(0) = x_0, \qquad y = h(x)$$

where $f(x)$ is locally Lipschitz, and $G(x)$, $h(x)$ are continuous over R^n. The matrix G is $n \times m$ and $h : R^n \to R^q$. The function f and h vanish at the origin; that is, $f(0) = 0$ and $h(0) = 0$. Let γ be a positive number and suppose there is a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the Hamilton-Jacobi (HJ) inequality

$$\mathcal{H}(V, f, G, h, \gamma) \triangleq \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

for all $x \in R^n$. For each $x_0 \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ.

Proof: Easy. Complete square for $\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u$, use HJ inequality, integrate, and work with the inequality. See proof in the book.

Theorem (Alternative to Theorem 5.4): Consider the linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

Suppose there is a positive semidefinite solution P of the Riccati equation

$$PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C = 0$$

for some $\gamma > 0$. Then, the system is $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ.

Proof: Trivial application of Theorem 5.5. See Example 5.11. It plays critical role in $\mathcal{H}_\infty$ Control $\rightarrow$ (Robust Control) synthesis.

Note: This theorem gives an alternative method for computing an upper bound on the $\mathcal{L}_2$ gain, as opposed to the frequency-domain calculation of Theorem 5.4.

Feedback Systems

Consider the linear time-invariant system

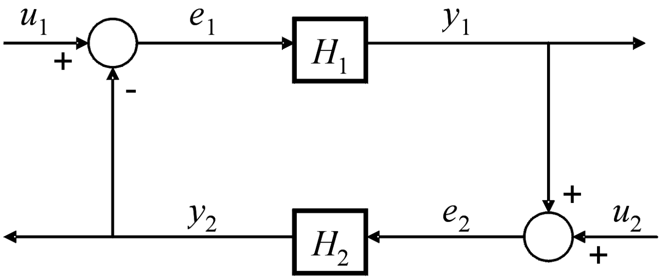


Figure: Feedback Connection.

Theorem 5.6 (Small-Gain Theorem): Suppose H_1 and H_2 are finite-gain $\mathcal{L}$ stable with constants (γ_1, β_1) and (γ_2, β_2) respectively. That is,

$$\begin{array}{lcllll} \|y_{1\tau}\|_{\mathcal{L}} & \leq & \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, & \forall e_1 \in \mathcal{L}_e^m, & \forall \tau \in [0, \infty) \\ \|y_{2\tau}\|_{\mathcal{L}} & \leq & \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, & \forall e_2 \in \mathcal{L}_e^q, & \forall \tau \in [0, \infty) \end{array}$$

Then, the feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1 \gamma_2 < 1$.

Proof: Assuming existence of solution, we can write

$$e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau, \qquad e_{2\tau} = u_{2\tau} - (H_1 e_1)_\tau.$$

Then,

$$\begin{array}{lcl} \|e_{1\tau}\|_{\mathcal{L}} & \leq & \|u_{1\tau}\|_{\mathcal{L}} + \|(H_2 e_2)_\tau\|_{\mathcal{L}} \\ & \leq & \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2 \\ & \leq & \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 (\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1) + \beta_2 \\ & = & \gamma_1 \gamma_2 \|e_{1\tau}\|_{\mathcal{L}} + (\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}} + \gamma_2 \beta_1 + \beta_2) \end{array}$$

Input-Output Stability

Since $\gamma_1\gamma_2 < 1$,

$$\|e_{1\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \gamma_2\beta_1 + \beta_2)$$

for all $\tau \in [0,\infty)$. Similarly,

$$\|e_{2\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\|u_{1\tau}\|_{\mathcal{L}} + \gamma_1\beta_2 + \beta_1)$$

for all $\tau \in [0,\infty)$.

Notes:

- Phase does NOT matter at all.
- Magnitude of one of the gains dos NOT matter at all as long as the other gain is small enough to satisfy $\gamma_1\gamma_2 < 1$.
- This theorem is the foundation of Robust Control (robust stability), where H_1 is seen as a stable nominal system and H_2 is seen as a stable perturbation.

Passivity

Passivity provides us with a useful tool for the analysis of nonlinear systems, which relates nicely to Lyapunov and $\mathcal{L}_2$ stability.

Definition 6.3: The system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

is said to be *passive* if there exists a continuously differentiable positive semidefinite function $V(x)$ (called the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall (x,u) \in R^n \times R^p$$

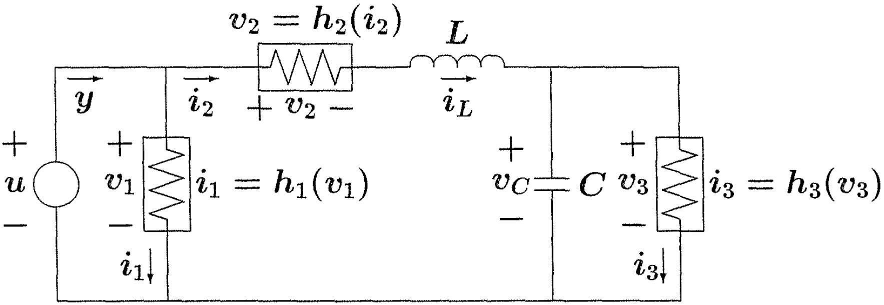
By integrating this equation, we can note that the system is passive if the energy (integral of the power $u^T y$) absorbed by the system over any period of time $[0,t]$ is greater than or equal to the increase in the energy stored in the system over the same period of time, that is

$$\int_0^t u(s)^T y(s) ds \geq V(x(t)) - V(x(0))$$

See Example 6.1 in the book.

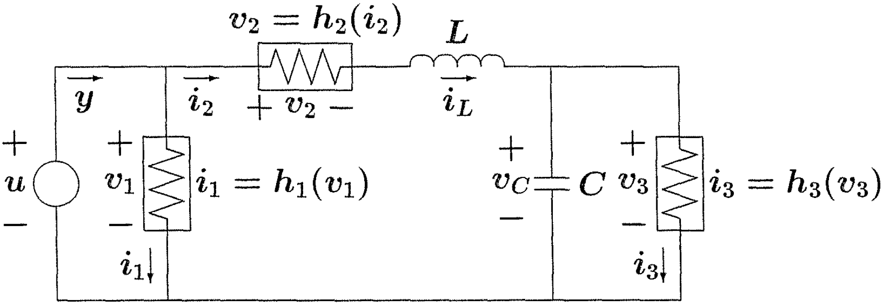
Passivity

Example: Khalil 6.1



- Voltage source connected to RLC network
- Linear inductor and capacitor; nonlinear resistors
- Resistors 1 and 3 represented by $v - i$ characteristics $i_1 = h_1(v_1)$, $i_3 = h_3(v_3)$
- Resistor 2 represented by $i - v$ characteristics $v_2 = h_2(i_2)$

Example: Khalil 6.1



Defining x_1 : current through inductor, x_2 : voltage across capacitor, and taking the voltage u as the input and the current y as the output, we can write

$$\begin{aligned} L\dot{x}_1 &= u - h_2(x_1) - x_2 \\ C\dot{x}_2 &= x_1 - h_3(x_2) \\ y &= x_1 + h_1(u) \end{aligned}$$

Passivity

Example: Khalil 6.1

The key feature of an RLC network over a resistive network is the the presence of the energy-storing elements L and C. The system is passive if the energy absorbed by the network over a period of time is greater than or equal to the increase in the energy stored in the network over the same period of time, i.e.

$$\int_0^t u(s)y(s)ds \geq V(x(t)) - V(x(0))$$

where $V(x) = (1/2)Lx_1^2 + (1/2)Cx_2^2$ is the energy stored in the network. Note that a strict inequality implies energy dissipation (resistors). Since this integral relationship must hold for every $t \geq 0$, the instantaneous power inequality

$$u(t)y(t) \geq \dot{V}(x(t))$$

must hold for all t. Note that $u(t)y(t)$ represents the power (voltage $\times$ current) flow into the network. Therefore, the power flow in the network must be greater than or equal to the rate of change of the energy stored in the network.

Example: Khalil 6.1

Let us investigate this inequality by computing the derivative of $V(x) = (1/2)Lx_1^2 + (1/2)Cx_2^2$ along the system trajectories, i.e.

$$\begin{aligned}\dot{V} = Lx_1\dot{x}_1 + Cx_2\dot{x}_2 &= x_1(u - h_2(x_1) - x_2) + x_2(x_1 - h_3(x_2)) \\ &= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\ &= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)) \\ &= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)\end{aligned}$$

Thus

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)$$

Example: Khalil 6.1

- If the memoryless functions h_1, h_2 and h_3 are passive ($h(u)u \geq 0$), then $uy \geq \dot{V}$ and the system is passive.
- if $h_1 = h_2 = h_3 = 0$, $uy = \dot{V}$ and the system is lossless (no energy dissipation).
- if h_2 and h_3 belong to the sector $[0, \infty]$ ($h(u)u \geq 0$), $uy \geq \dot{V} + uh_1(u)$. The term $uh_1(u)$ represents excess (> 0) or shortage (< 0) of passivity. This type of excess or shortage of passivity can be removed by input feedforward. In the case of excess of passivity we are talking about input strict passivity.
- if $h_1 = 0$ and h_3 belong to the sector $[0, \infty]$ ($h(u)u \geq 0$), $uy \geq \dot{V} + yh_2(y)$. The term $yh_2(y)$ represents excess (> 0) or shortage (< 0) of passivity. This type of excess or shortage of passivity can be removed by output feedback. In the case of excess of passivity we are talking about output strict passivity.
- if h_1 and h_2 belong to the sector $[0, \infty]$ ($h(u)u \geq 0$), and h_3 belongs to the sector $(0, \infty)$ ($h(u)u > 0$) $uy \geq \dot{V} + x_1h_2(x_1) + x_2h_3(x_2)$. The term $x_1h_2(x_1) + x_2h_3(x_2)$ is a positive definite function of x and represents an excess of passivity. In the case of excess of passivity we are talking about state strict passivity.

Passivity

Therefore, the system is said to be passive if $u^T y \geq \dot{V}$. Moreover, it is said to be

- *lossless* if $u^T y = \dot{V}$.
- *input-feedforward passive* if $u^T y \geq \dot{V} + u^T \phi(u)$ for some function ϕ.
- *input strictly passive* if $u^T y \geq \dot{V} + u^T \phi(u)$ and $u^T \phi(u) > 0$, for all $u \neq 0$.
- *output-feedback passive* if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ.
- *output strictly passive* if $u^T y \geq \dot{V} + y^T \rho(y)$ and $y^T \rho(y) > 0$, for all $y \neq 0$.
- *strictly passive* if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ.

In all cases the inequality should hold for all (x, u).

Passivity

It is useful to write the passivity condition as

$$\dot{V} \leq -\rho\psi(x) - \delta y^T y - \epsilon u^T u + u^T y$$

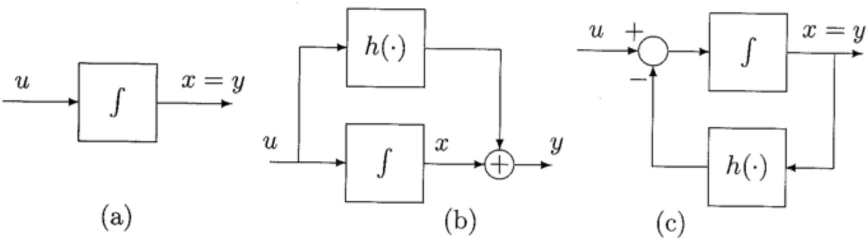
The term $u^T y$ contains the phase information. The system is:

- *lossless* if $\rho = \delta = \epsilon = 0$, i.e., $\dot{V} \leq u^T y$
- *input strictly passive* if $\epsilon > 0$
- *output strictly passive* if $\delta > 0$
- *state strictly passive* if $\rho > 0$

Note: V is allowed to be positive semidefinite.

Passivity

Example: Khalil 6.2.



Relationship with $\mathcal{L}_2$ stability:

Lemma 6.5: If the system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

with $f(0,0) = h(0,0) = 0$ is output strictly passive with $u^T y \geq \dot{V} + \delta y^T y$, for some $\delta > 0$, then it is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\delta$.

Proof: The derivative of the storage function $V(x)$ satisfies

$$\begin{aligned} \dot{V} &\leq u^T y - \delta y^T y \\ &= -\frac{1}{2\delta}(u - \delta y)^T (u - \delta y) + \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \\ &\leq \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \end{aligned}$$

Integrating both sides over $[0, \tau]$ yields

$$\int_0^\tau y(t)^T y(t) dt \leq \frac{1}{\delta^2} \int_0^\tau u^T(t) u(t) dt - \frac{2}{\delta}[V(x(\tau)) - V(x(0))]$$

Thus,

$$\|y_\tau\|_{\mathcal{L}_2} \leq \frac{1}{\delta} \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\delta} V(x(0))}$$

where we have used the fact that $V(x) \geq 0$ and $\sqrt{a^2 + b^2} \leq a + b$ for non-negative numbers a and b.

Relationship with Lyapunov stability:

Lemma 6.6: If the system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

with $f(0,0) = h(0,0) = 0$ is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x} = f(x,0)$ is stable.

Proof: Take V as Lyapunov function candidate for $\dot{x} = f(x,0)$. Then $\dot{V} \leq 0$.

Note: To show asymptotic stability of the origin of $\dot{x} = f(x,0)$, we need to either show that $\dot{V}$ is negative definite or apply the invariance theorem. The next lemma uses the invariance theorem in combination with an additional property of the system to give conditions for asymptotic stability.

Definition 6.5: The system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

with $f(0,0) = h(0,0) = 0$ is said to be *zero-state observable* if no solution of $\dot{x} = f(x,0)$ can stay identically in $S = \{x \in R^n | h(x,0) = 0\}$, other than $x(t) = 0$.

Lemma 6.7: Consider the system $\dot{x} = f(x,u)$, $y = h(x,u)$. The origin of $\dot{x} = f(x,0)$ is asymptotically stable if system is

- strictly passive or
- output strictly passive and zero-state observable

If the storage function is radially unbounded, the origin will be *globally asymptotically stable*.

Proof: See proof in the book.

Passivity

Example: Khalil 6.6.

Feedback Systems:

Consider the linear time-invariant system

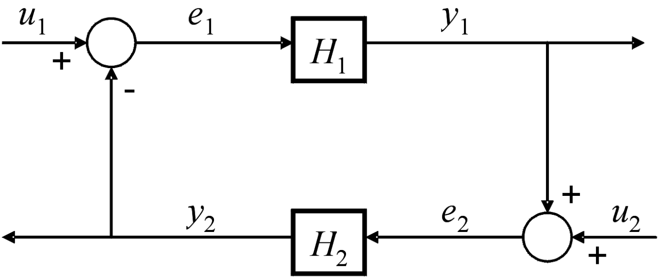


Figure: Feedback Connection.

Each system H_i can be represented by a state-space model

$$\begin{aligned} \dot{x}_i &= f_i(x_i, e_i) \\ y_i &= h_i(x_i, e_i) \end{aligned}$$

The closed-loop state model is

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

with $f(0,0) = h(0,0) = 0$ (assumption), where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Theorem 6.1: The feedback connection of two passive systems is passive.

Proof: See proof in the book. Very easy. Results are derived from definition of passivity of individual systems, feedback connection, and taking $V = V_1 + V_2$, where V_1 and V_2 are storage functions of H_1 and H_2.

Lemma 6.8: The feedback connection of two output strictly passive systems with

$$e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i, \qquad \delta_i > 0$$

is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\min\{\delta_1, \delta_2\}$.

Proof: See proof in the book. Very easy. Take once again $V = V_1 + V_2$, where V_1 and V_2 are storage functions of H_1 and H_2, and $\delta = \min\{\delta_1, \delta_2\}$.

Theorem 6.2: Consider the feedback connection of two time-invariant dynamical systems that satisfy

$$e_i^T y_i \geq \dot{V}_i + \epsilon_i e_i^T e_i + \delta_i y_i^T y_i, \qquad \epsilon_i, \delta_i > 0$$

for some storage function $V_i(x_i)$. Then, the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable if $\epsilon_1 + \delta_2 > 0$, $\epsilon_2 + \delta_1 > 0$.

Proof: See book. Note that Theorem 6.2 reduces to Lemma 6.8 when $\epsilon_i \equiv 0$.

Theorem 6.3: Consider the feedback connection of two time-invariant dynamical systems:

$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i), \qquad i = 1, 2$$

The origin of the close loop system (when $u{=}0$) is asymptotically stable if

- both feedback components are strictly passive
- both feedback components are output strictly passive and zero-state observable
- one component is strictly passive and the other one is output strictly passive and zero state observable

Moreover, if storage function of each component is radially unbounded, the origin is *globally asymptotically stable*.

Proof: Similar to Lemma 6.7. See book.

Passivity

ISS:

$$\dot{V} \leq -\alpha\left(|x|\right) + \rho\left(|u|\right)$$

SS Pasive:

$$\dot{V} \leq -\alpha\left(|x|\right) + u^T y$$

"Neither one implies the other"

Feedback Interconnections:

- Small gain: Two stable systems that do not increase *amplitude* remains stable
- Passivity: Two stable systems that do not invert *phase (sign)* remains stable

Passivity-Based Control

We consider the p-input-p-output system

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned}$$

We assume that $f(0,0)=0$ so that the origin is an open-loop equilibrium, and $h(0)=0$. The system is said to be *passive* if there exists a continuously differentiable positive semidefinite function $V(x)$ (called the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall (x,u) \in R^n \times R^p$$

The system is *zero-state observable* if no solution of $\dot{x} = f(x,0)$ can stay identically in the set $\{h(x)=0\}$ other than the trivial solution.

Theorem 14.4: If the $p \times p$ system $\dot{x} = f(x, u)$, $y = h(x)$ is

- passive with a radially unbounded positive definite storage function $V(x)$ and
- zero-state observable

the the origin $x = 0$ can be globally stabilized by $u = -\phi(y)$, where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y^T \phi(y) > 0$ for all $y \neq 0$.

Proof: Use the storage functio0n $V(x)$ as Lyapunov function candidate for the closed-loop system $\dot{x} = f(x, -\phi(y))$. The derivative of V is given by

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq y^T u = -y^T \phi(y) \leq 0$$

Hence, $\dot{V}$ is negative semidefinite and $\dot{V} = 0$ if and only if $y = 0$. By zero-state observability,

$$y(t) \equiv 0 \Rightarrow u(t) \equiv 0 \Rightarrow x(t) \equiv 0$$

Then, by the Invariance Principle, the origin is globally asymptotically stable.

Note: Intuition behind theorem becomes clear when we think the storage function as energy of system. A passive system has a stable origin. All that is needed to stabilize origin is the injection of damping so that energy will dissipate whenever $x(t)$ is different from zero. Required damping is injected by function ϕ (freedom in the choice).

Passivity-Based Control

The utility of Theorem 14.4 can be increased by transforming nonpassive systems into passive ones. We consider a special case of $\dot{x} = f(x, u)$, where

$$\dot{x} \quad = \quad f(x) + G(x)u$$

- Suppose a radially unbounded, positive definite, continuously differentiable function $V(x)$ exists such that

  $$\frac{\partial V}{\partial x} f(x) \leq 0, \quad \forall x \quad \text{(origin is open-loop stable)}$$

  If we take

  $$y = h(x) = \left[\frac{\partial V}{\partial x} G(x) \right]^T,$$

  the system with input u and output y is passive. If it is also zero-state observable, we can apply Theorem 14.4

Example: Khalil 14.15

Passivity-Based Control

- Allowing ourselves the freedom to choose the output function is useful, but we are still limited to state equations for which the origin is open-loop stable. If a feedback control

$$u = \alpha(x) + \beta(x)v$$

and a function $h(x)$ exist such that the system

$$\begin{aligned} \dot{x} &= f(x) + G(x)\alpha(x) + G(x)\beta(x)v \\ y &= h(x) \end{aligned}$$

with input v and output y, satisfies the conditions of Theorem 14.4, we can globally stabilize the origin by using $v = -\phi(y)$. The use of feedback to convert a nonpassive system into a passive one is known as *feedback passivation*.

Passivity-Based Control

The cascade connection of a passive system with a system whose unforced dynamics has a stable equilibrium at the origin is amenable to feedback passivation. Let us consider

$$\dot{z} = f_a(z) + F(z,y)y \tag{1}$$

$$\dot{x} = f(x) + G(x)u \tag{2}$$

$$y = h(x) \tag{3}$$

- If the driving-system (2)-(3) is passive with a radially unbounded positive definite storage function $V(x)$, and if the origin of the driven-system (1) with $y = 0$, i.e., the origin of $\dot{z} = f_a(z)$ is stable with a radially unbounded Lyapunov function $W(z)$ that satisfies

$$\frac{\partial W}{\partial z} f_a(z) \leq 0 \quad \forall z$$

Then, the feedback control

$$u = -\left(\frac{\partial W}{\partial z} F(z, y)\right)^T + v$$

makes the system

$$\begin{aligned} \dot{z} &= f_a(z) + F(z, y)y && (4) \\ \dot{x} &= f(x) - G(x)\left(\frac{\partial W}{\partial z} F(z, y)\right)^T + G(x)v && (5) \\ y &= h(x) && (6) \end{aligned}$$

with input v, output y is passive with $U(z, x) = W(z) + V(x)$ as the storage function. If the system (4)-(6) is zero-state observable, we can apply Theorem 14.4 to globally stabilize the origin.

Passivity-Based Control

- Checking zero-state observability of the system (4)-(6) can be avoided by strengthening the assumption on $W(z)$. If the driving-system (2)-(3) is zero-state observable and passive with a radially unbounded positive definite storage function $V(x)$, and if the origin of the driven-system (1) with $y = 0$, i.e., the origin of $\dot{z} = f_a(z)$, is globally asymptotically stable with a radially unbounded Lyapunov function $W(z)$ that satisfies

$$\frac{\partial W}{\partial z} f_a(z) < 0 \quad \forall z \neq 0, \qquad \frac{\partial W}{\partial z}(0) = 0.$$

The feedback control

$$u = -\left(\frac{\partial W}{\partial z} F(z, y) \right)^T - \phi(y),\ \phi(0) = 0,\ y^T \phi(y) > 0\ \forall y \neq 0$$

globally stabilizes the origin $(z = 0, x = 0)$.

Example: Khalil 14.18