Nonlinear Systems and Control

Lecture 7 (Meetings 23-25)
Nonlinear Controllability
Chapter 14: Backstepping - Lyapunov Redesign

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NOTE: This material is not in Khalil's book.

Let us focus on *driftless* systems:

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m$$

Definition: The system is *completely controllable* if given any $T > 0$ and any pair of points $x_0, x_1 \in R^n$ there is an input $u = (u_1, \ldots, u_m)$ which is piecewise analytic on $[0, T]$ and which steers the system from $x(0) = x_0$ to $x(T) = x_1$.

Note: Recall that

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$$

Nonlinear Controllability

Chow's Theorem: The system is *completely controllable* if and only if $g_1, \ldots, g_m$ plus all repeated Lie brackets span every direction.

Note: If we have only one control u_1, then $[g_1, g_1] = 0$. Thus, we cannot generate any other direction. Here we need several controls u_i.

Note: The involutive closure of a distribution Δ is the closure $\bar{\Delta}$ of the distribution under Lie bracketing.

- Given a distribution take all Lie brackets
- If you get new vector fields, add them to the distribution
- Repeat until you get no new vector field

Nonlinear Controllability

Control Lie Algebra:

$$\mathcal{C} = \{x | x = [x_j, [x_{j-1}, [\ldots [x_1, x_0]]]]\}$$

$$x_i \in g_1, \ldots, g_m, \quad i = 1, \ldots j, \quad j = 1, 2, \ldots$$

$$\begin{aligned} \mathcal{L} &= \text{span}\,\{\mathcal{C}\} \\ &= \text{span}\,\{g_1, \ldots, g_m, [g_1, g_2], [g_1, g3], \ldots, [g_1, [g_2, g_3], \ldots\} \end{aligned}$$

Chow's Theorem: The system is *completely controllable* if and only if $\dim \mathcal{L}(x) = n$ for all x.

Chow's Theorem: The system is *completely controllable* if and only if the involutive closure of $\{g_1, \ldots, g_m\}$ is of constant rank n for all x.

Nonlinear Controllability

Consider now the systems:

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m$$

Chow's Theorem: The system is *completely controllable* if and only if the involutive closure of $\{f, g_1, \ldots, g_m\}$ is of constant rank n for all x.

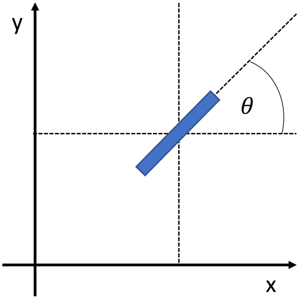
For example, for $m = 1$, if the system is input-state linearizable, then it is completely controllable

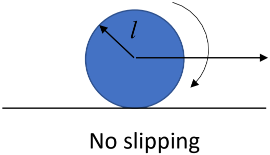
completely controllable $\not\Rightarrow$ input-state linearizable

Example 1: Unicycle

$$\begin{aligned} \dot{x} &= l\cos(\theta)u_1 \\ \dot{y} &= l\sin(\theta)u_1 \\ \dot{\theta} &= u_2 \end{aligned}$$

It is *completely controllable*





Nonlinear Controllability

Example 1: Unicycle

Linearization at $x = y = \theta = 0$:

$$\begin{aligned} \dot{x} &= lu_1 \\ \dot{y} &= 0 \\ \dot{\theta} &= u_2 \end{aligned}$$

It is NOT controllable! $\Rightarrow$ completely controllable due to nonlinearity!

Nonlinear Stabilizability

Linear systems: Controllability $\Rightarrow$ Stabilizability
Nonlinear systems: Controllability $\not\Rightarrow$ Stabilizability

Brockett's Theorem: If the equilibrium $x = 0$ of the C^1 system $\dot{x} = f(x, u)$ is locally asymptotically stabilizable by C^1 feedback of x, then ($\Rightarrow$) the image of the mapping $f(x, u)$ contains some neighborhood of $x = 0$, i.e., $\exists \delta > 0$ such that $\forall |\xi| \leq \delta \; \exists x, u$ such that $f(x, u) = \xi$.

Reminiscent of the Hautus-Popov-Belevitch Controllability Test

$$\begin{aligned} \text{rank}[sI - A, B] &= n \quad \forall s \\ \text{image}[sI - A, B] &= R^n \quad \forall s \end{aligned}$$

Nonlinear Stabilizability

Example 2: Unicycle

$$\begin{aligned}\dot{x} &= l\cos(\theta)u_1\\ \dot{y} &= l\sin(\theta)u_1\\ \dot{\theta} &= u_2\end{aligned}$$

on the set $|\theta| < \pi/2$.

It is not stabilizable by C^1 feedback!

Nonlinear Stabilizability

Brockett's Theorem:

- Only necessary condition
- Restricted to C^1 feedback of x

There are two possibilities for systems that violate Brockett's condition:

- Non-smooth feedback
- Time-varying feedback

Nonlinear Stabilizability

Example 3: Unicycle

$$\begin{aligned} \dot{x} &= l\cos(\theta)u_1 \\ \dot{y} &= l\sin(\theta)u_1 \\ \dot{\theta} &= u_2 \end{aligned}$$

Control Lyapunov Function (CLF)

We are interested in an extension of the Lyapunov function concept, called a *control Lyapunov function* (CLF).

Let us consider the following system:

$$\dot{x} = f(x, u), \quad x \in R^n, \quad u \in R, \quad f(0, 0) = 0,$$

Task: Find a feedback control law $u = \alpha(x)$ such that the equilibrium $x = 0$ of the closed-loop system

$$\dot{x} = f(x, \alpha(x))$$

is globally asymptotically stable.

Task: Find a feedback control law $u = \alpha(x)$ and a Lyapunov function candidate $V(x)$ such that

$$\dot{V} = \frac{\partial V}{\partial x}(x) f(x, \alpha(x)) \leq -W(x), \quad W(x) \text{ positive definite}$$

A system for which a good choice of $V(x)$ and $W(x)$ exists is said to possess a CLF.

Control Lyapunov Function (CLF)

Definition: A smooth positive definite and radially unbounded function $V : R^n \rightarrow R_+$ is called a control Lyapunov function (CLF) if

$$\inf_{u \in R} \{ \frac{\partial V}{\partial x}(x) f(x,u) \} < 0 \qquad \forall x \neq 0$$

$$\text{(or } \forall x \, \exists u \text{ s.t. } \frac{\partial V}{\partial x}(x) f(x,u) < 0)$$

Control Lyapunov Function (CLF)

Let us consider the following system affine in control:

$$\dot{x} = f(x) + g(x)u, \quad x \in R^n, \quad u \in R, \quad f(0) = 0,$$

Definition: A smooth positive definite and radially unbounded function $V : R^n \to R_+$ is called a control Lyapunov function (CLF) if $\forall x \; \exists u$ such that

$$\frac{\partial V}{\partial x}(x) f(x) + \frac{\partial V}{\partial x}(x) g(x) u < 0 \quad \forall x \neq 0$$

So, $V(x)$ must satisfy (equivalent)

$$\frac{\partial V}{\partial x}(x) g(x) = 0 \Rightarrow \frac{\partial V}{\partial x}(x) f(x) < 0 \quad \forall x \neq 0$$

The "uncontrollable" part is stable by itself.

Control Lyapunov Function (CLF)

Artstein ('83): If a CLF exists, then $\alpha(x)$ exists (but the proof is not constructive)

Naive formula: (not continuous at $\frac{\partial V}{\partial x}(x)g(x) = 0$)

$$u = \alpha(x) = -\frac{\frac{\partial V}{\partial x}(x)f(x) + W(x)}{\frac{\partial V}{\partial x}(x)g(x)}$$

Sontag's formula ('89):

$$u = \alpha_s(x) = \begin{cases} -\frac{\frac{\partial V}{\partial x}(x)f(x) + \sqrt{\left(\frac{\partial V}{\partial x}(x)f(x)\right)^2 + \left(\frac{\partial V}{\partial x}(x)g(x)\right)^4}}{\frac{\partial V}{\partial x}(x)g(x)} & \frac{\partial V}{\partial x}(x)g(x) \neq 0 \\ 0 & \frac{\partial V}{\partial x}(x)g(x) = 0 \end{cases}$$

This control gives $\dot{V} = -\sqrt{\left(\frac{\partial V}{\partial x}(x)f(x)\right)^2 + \left(\frac{\partial V}{\partial x}(x)g(x)\right)^4} < 0$

Control Lyapunov Function (CLF)

Question: Is $\alpha_s(x)$ continuous on R^n?

Lemma: $\alpha_s(x)$ is smooth on R^n.

Lemma: $\alpha_s(x)$ is continuous at $x = 0$ *if and only if* the CLF satisfies the *small control property*: $\forall \epsilon$, $\exists \delta(\epsilon) > 0$ such that if $|x| < \delta$, $\exists |u| < \epsilon$ such that

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)u] < 0$$

In other words, if there is a continuous controller stabilizing $x = 0$ w.r.t. the given V, then $\alpha_s(x)$ is also continuous at zero.

Sontag's formula is continuous at the origin and smooth away from the origin

Control Lyapunov Function (CLF)

Theorem: A system is stabilizable *if and ony if* there exists a CLF

Proof:

- There is a CLF $\Rightarrow$ system is stabilizable (proved)
- System is stabilizable $\Rightarrow$ there is a CLF (Converse Lyapunov theorem)

Control Lyapunov Function (CLF)

Example 4:

$$\dot{x} = -x^3 + u$$

Example 5:

$$\dot{x} = x^3 + x^2 u$$

Backstepping

Let us consider the following system affine in control:

$$\dot{x} = f(x) + g(x)u, \quad x \in R^n, \quad u \in R, \quad f(0) = 0,$$

Assumption: There exist $u = \alpha(x)$ and $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\alpha(x)] \leq -W(x), \quad W(x) \text{ positive definite}$$

Backstepping

Lemma: Integrator Backstepping

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u \end{aligned}$$

There is a whole integrator between u and ξ. Under the previous assumption, the system has a CLF

$$V_a(x, \xi) = V(x) + \frac{1}{2}(\xi - \alpha(x))^2, \qquad \text{(a: augmented)}$$

and the corresponding feedback that gives global asymptotical stability is

$$u = -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi] - \frac{\partial V}{\partial x} g(x), \qquad c > 0$$

Backstepping: We have a "virtual" control ξ and we have to go back through an integrator.

Backstepping

Proof: In class.

Example 6: Avoid singularities in feedback linearization

$$\begin{aligned}\dot{x} &= x\xi \\ \dot{\xi} &= u\end{aligned}$$

Backstepping

In the case of more than one integrator

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n \\
\dot{\xi}_n &= u
\end{aligned}
$$

we only have to apply the backstepping lemma n times.

Example 7: Khalil Examples 14.8

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= u\end{aligned}$$

Example 8: Khalil Examples 14.9

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned}$$

Backstepping

In the more general case

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= f_a(x,\xi) + g_a(x,\xi)u \end{aligned}$$

If $g_a(x,\xi) \neq 0$ over the domain of interest, the input transformation

$$u = \frac{1}{g_a(x,\xi)}[v - f_a(x,\xi)]$$

will reduce the system to

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= v \end{aligned}$$

and the backstepping lemma can be applied.

Backstepping

Strict Feedback Systems: By recursive application of backstepping, we can stabilize strict-feedback systems of the form

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= u + \phi_n(x) \end{aligned}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T$, $\phi_i(\bar{x}_i)$ are smooth and $\phi_i(0) = 0$.
We have a local triangular structure:

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(x_1) \\ \dot{x}_2 &= x_3 + \phi_2(x_1, x_2) \\ &\vdots \\ \dot{x}_n &= u + \phi_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

Linear part: Brunovsky canonical form $\Rightarrow$ feedback linearizable

Backstepping

The control law

$$
\begin{aligned}
z_i &= x_i - \alpha_{i-1}(\bar{x}_{i-1}) \quad \alpha_0 = 0 \\
\alpha_i(\bar{x}_i) &= -z_{i-1} - c_i z_i - \phi_i + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j), \quad c_i > 0 \\
u &= \alpha_n
\end{aligned}
$$

guarantees global asymptotic stability of $x = 0$.

Proof: In class.

Backstepping

The technique can be extended to more general **Strict Feedback Systems:**

$$\begin{aligned} \dot{x}_i &= \psi_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= \psi_n(x)u + \phi_n(x) \end{aligned}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T$ ($\bar{x}_n = x$), $\phi_i(\bar{x}_i)$ are smooth and $\phi_i(0) = 0$, and $\psi_i(\bar{x}_i) \neq 0$ for $i = 1, \ldots, n$ over the domain of interest.

Assumption: There exist $\xi = \phi(\eta)$ with $\phi(0) = 0$, and $V(x)$ such that

$$\frac{\partial V}{\partial \eta}[f(\eta) + G(\eta)\phi(\eta)] \leq -W(\eta), \quad W(\eta) \text{ positive definite}$$

Lemma: Block Backstepping

$$\begin{aligned} \dot{\eta} &= f(\eta) + G(\eta)\xi \\ \dot{\xi} &= f_a(\eta, \xi) + G_a(\eta, \xi)u \end{aligned}$$

where $\eta \in R^n$, $\xi \in R^m$, and $u \in R^m$, in which m can be greater than one. Under the previous assumption, the system has a CLF

$$V_c(\eta, \xi) = V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^T[\xi - \phi(\eta)],$$

and the corresponding feedback that gives asymptotical stability for the equilibrium at the origin is

$$u = G_a^{-1}\left[\frac{\partial \phi}{\partial \eta}[f(\eta) + G(\eta)\xi] - \left(\frac{\partial V}{\partial \eta}G(\eta)\right)^T - f_a - k(\xi - \phi(\eta))\right]$$

with $k > 0$.

Proof: Check the book.

Backstepping

Robust Control: Consider the system

$$\begin{aligned}\dot{\eta} &= f(\eta) + g(\eta)\xi + \delta_n(\eta, \xi) \\ \dot{\xi} &= f_a(\eta, \xi) + g_a(\eta, \xi)u + \delta_\xi(\eta, \xi)\end{aligned}$$

where $\eta \in R^n$, $\xi \in R$, and $g_a(\eta, \xi) \neq 0$. The uncertainty terms δ_n and δ_ξ satisfy inequalities

$$\begin{aligned}\|\delta_n(\eta, \xi)\|_2 &\leq a_1\|\eta\|_2 \\ |\delta_\xi(\eta, \xi)| &\leq a_2\|\eta\|_2 + a_3|\xi|\end{aligned}$$

Let $\xi = \phi(\eta)$ with $\phi(0) = 0$ be a stabilizing state feedback control law for the η-system that satisfies

$$|\phi(\eta)| \leq a_4\|\eta\|_2, \qquad \left\|\frac{\partial \phi}{\partial \eta}\right\|_2 \leq a_5$$

Backstepping

and $V(\eta)$ be a Lyapunov function that satisfies

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + \delta_n(\eta, \xi)] \leq -b\|\eta\|_2^2$$

Then, the state feedback control law

$$u = \frac{1}{g_a}\left[\frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta} g(\eta) - f_a - k(\xi - \phi)\right]$$

with k sufficiently large, stabilizes the origin of our system. Moreover, if all assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be globally asymptotically stable.

Proof: Take $V_c(\eta, \xi) = V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2$. Check the book.

Example 9: Avoid cancellation

$$\begin{aligned}\dot{x} &= x - x^3 + \xi \\ \dot{\xi} &= u\end{aligned}$$

Lyapunov Redesign

Stabilization: Let us consider the following system:

$$\dot{x} = f(t,x) + G(t,x)\left[u + \delta(t,x,u)\right], \quad x \in R^n, \quad u \in R^p.$$

The uncertain term δ is an unknown function that lumps together various uncertain terms due to model simplification, parameter uncertainty, etc. The uncertain term δ satisfies the *matching condition*, i.e., the uncertain term δ enters the state equation at the same point as the control input u.

Suppose we designed a control law $u = \psi(t,x)$ such that the origin of the nominal closed loop system

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x)$$

is uniformly asymptotically stable.

Lyapunov Redesign

Suppose further that we know a Lyapunov function $V(t,x)$ that satisfies

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left[f(t,x) + G(t,x)\psi(t,x)\right] \leq -\alpha_3(\|x\|)$$

for all $t \geq 0$ and for all $x \in D$, where α_1, α_2 and α_3 are class $\mathcal{K}$ functions.

We assume that, with $u = \psi(t,x) + v$, the uncertainty term δ satisfies the inequality

$$\|\delta(t,x,\psi(t,x)+v)\| \leq \rho(t,x) + k_0\|v\| \quad \rho \geq 0, 0 \leq k_0 < 1$$

GOAL: Design v s.t. overall control $u = \psi(t,x) + v$ stabilizes the actual system in the presence of the uncertainty.

Lyapunov Redesign

Let us apply the control law $u = \psi(t,x) + v$ to our original system, i.e.

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x) + G(t,x)\left[v + \delta(t,x,\psi(t,x)+v)\right]$$

And let us calculate the derivative along its trajectories, i.e.

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + G\psi) + \frac{\partial V}{\partial x}G(v+\delta) \leq -\alpha_3(\|x\|) + \frac{\partial V}{\partial x}G(v+\delta)$$

Let us set $w^T \triangleq \frac{\partial V}{\partial x}G$ to rewrite the inequality as

$$\dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T \delta$$

Due to matching condition, δ enters the equation in the same way v does. Therefore, it is possible to choose v to cancel the destabilizing effect of δ.

Solutions: Non-Continuous

$$\begin{array}{lcl} \|\delta(t,x,\psi(t,x)+v)\|_2 \leq \rho(t,x) + k_0\|v\|_2 & \rightarrow & v = -\eta(t,x)\frac{w}{\|w\|_2} \\ \|\delta(t,x,\psi(t,x)+v)\|_\infty \leq \rho(t,x) + k_0\|v\|_\infty & \rightarrow & v = -\eta(t,x)\mathsf{sgn}(w) \end{array}$$

where

$$w^T = \frac{\partial V}{\partial x}G, \quad \eta(t,x) \geq \rho(t,x)/(1-k_0) \quad \forall (t,x)$$

Note that (inequality satisfied with $\|\cdot\| = \|\cdot\|_2$)

$$\begin{aligned} \dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T\delta \quad &\leq \quad -\alpha_3(\|x\|) + w^T v + \|w\|(\rho(t,x) + k_0\|v\|) \\ &\leq \quad -\alpha_3(\|x\|) - \eta\|w\| + \|w\|\rho(t,x) + k_0\eta\|w\| \\ &\leq \quad -\alpha_3(\|x\|) - (\eta(1-k_0) - \rho(t,x))\|w\| \\ &\leq \quad -\alpha_3(\|x\|) \end{aligned}$$

Lyapunov Redesign

Note that (inequality satisfied with $\|\cdot\| = \|\cdot\|_\infty$)

$$\begin{aligned}\dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T \delta \quad &\leq \quad -\alpha_3(\|x\|) + w^T v + \|w\|_1(\rho(t,x) + k_0\|v\|_\infty) \\ &\leq \quad -\alpha_3(\|x\|) - \eta\|w\|_1 + \|w\|_1\rho(t,x) + k_0\eta\|w\|_1 \\ &\leq \quad -\alpha_3(\|x\|) - (\eta(1-k_0) - \rho(t,x))\|w\|_1 \\ &\leq \quad -\alpha_3(\|x\|)\end{aligned}$$

These control laws are discontinuous functions of state x:

- Division by zero $\Rightarrow$ Control law needs to be redefined
- Not locally Lipschitz $\Rightarrow$ Solution existence/uniqueness?
- Chattering (fast switching fluctuations)

Note: In both cases we used Holder's inequality

$$|x^T y| \leq \|x\|_p \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Lyapunov Redesign

Solutions: Continuous (just for one of the controllers: $\|\cdot\| = \|\cdot\|_2$)

$$v = \begin{cases} -\eta(t,x)\frac{w}{\|w\|_2} & \text{if } \eta(t,x)\|w\|_2 \geq \epsilon \\ -\eta^2(t,x)\frac{w}{\epsilon} & \text{if } \eta(t,x)\|w\|_2 < \epsilon \end{cases}$$

We already showed that $\dot{V} < 0$ for $\eta(t,x)\|w\|_2 \geq \epsilon$. We need now to check $\dot{V}$ for $\eta(t,x)\|w\|_2 < \epsilon$:

$$\begin{aligned} \dot{V} \leq -\alpha_3(\|x\|) + w^T v + w^T \delta \quad & \leq \quad -\alpha_3(\|x\|) + w^T[-\eta^2 \frac{w}{\epsilon} + \delta] \\ & \leq \quad -\alpha_3(\|x\|) - \frac{\eta^2}{\epsilon}\|w\|^2 + \rho\|w\| + k_0\|w\|\|v\| \\ & = \quad -\alpha_3(\|x\|) - \frac{\eta^2}{\epsilon}\|w\|^2 + \rho\|w\| + k_0\frac{\eta^2}{\epsilon}\|w\|^2 \\ & \leq \quad -\alpha_3(\|x\|) + (1-k_0)\left(-\frac{\eta^2}{\epsilon}\|w\|^2 + \eta\|w\|\right) \end{aligned}$$

where we have used $\|\delta\| \leq \rho + k_0\|v\|$ and $\eta \geq \rho/(1-k_0) \iff \rho \leq (1-k_0)\eta$.

Lyapunov Redesign

The term $-\frac{\eta^2}{\epsilon}\|w\|^2 + \eta\|w\|$ attains a maximum value $\epsilon/4$ at $\eta\|w\| = \epsilon/2$. Then,

$$\dot{V} \quad \leq \quad -\alpha_3(\|x\|) + \frac{\epsilon(1-k_0)}{4}$$

whenever $\eta(t,x)\|w\|_2 < \epsilon$. On the other hand, when $\eta(t,x)\|w\|_2 \geq \epsilon$ we have

$$\dot{V} \quad \leq \quad -\alpha_3(\|x\|) \leq -\alpha_3(\|x\|) + \frac{\epsilon(1-k_0)}{4}$$

Then, this inequality is satisfied irrespective of the value of $\eta(t,x)\|w\|_2$. Choosing $\epsilon < 2\alpha_3(\alpha_2^{-1}(\alpha_1(r)))/(1-k_0)$, setting $\mu = \alpha_3^{-1}(\epsilon(1-k_0)/2) < \alpha_2^{-1}(\alpha_1(r))$, and taking $r > 0$ such that $B_r \subset D$. Then,

$$\dot{V} \leq -\frac{1}{2}\alpha_3(\|x\|_2), \quad \forall \mu \leq \|x\|_2 < r$$

- In general, continuous Lyapunov redesign does NOT stabilize origin as its discontinuous counterpart does
- However, it guarantees boundedness of the solution
- Stabilizes origin if uncertainty vanishes at origin and if

$$\alpha_3(\|x\|_2) \geq \phi^2(x), \quad \eta(t,x) \geq \eta_0 > 0, \quad \rho(t,x) \leq \rho_1\phi(x)$$

Lyapunov Redesign

Nonlinear Damping: Let us consider again the system:

$$\dot{x} = f(t,x) + G(t,x)\left[u + \delta(t,x,u)\right], \quad x \in R^n, \quad u \in R^p.$$

but with $\delta(t,x,u) = \Gamma(t,x)\delta_0(t,x,u)$, where δ_0 is a uniformly bounded uncertain term. The function $\Gamma(t,x)$ is known.

Suppose we designed a control law $u = \psi(t,x)$ such that the origin of the nominal closed loop system

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x)$$

is uniformly asymptotically stable.

GOAL: Design v s.t. overall control $u = \psi(t,x) + v$ stabilizes the actual system in the presence of the uncertainty.

Lyapunov Redesign

Suppose further that we know a Lyapunov function $V(t,x)$ that satisfies

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left[f(t,x) + G(t,x)\psi(t,x)\right] \leq -\alpha_3(\|x\|)$$

for all $t \geq 0$ and for all $x \in D$, where α_1, α_2 and α_3 are class $\mathcal{K}_\infty$ functions.

- If upper bound of δ_0 is known $\rightarrow$ *stabilization* as before (Lyapunov redesign).
- If upper bound of δ_0 is NOT known $\rightarrow$ *nonlinear damping* guarantees boundedness if

$$v = -kw\|\Gamma(t,x)\|_2^2, \qquad k > 0$$

Note that

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\left[f + G\psi\right] + \frac{\partial V}{\partial x}G\left[v + \Gamma\delta_0\right] \leq -\alpha_3(\|x\|) + w^T(v + \Gamma\delta_0)$$

where $w^T = \frac{\partial V}{\partial x}G$.

Lyapunov Redesign

Taking $v = -kw\|\Gamma(t,x)\|^2$, we obtain ($\|\cdot\| = \|\cdot\|_2$)

$$\dot{V} \leq -\alpha_3(\|x\|) - k\|w\|^2\|\Gamma(t,x)\|^2 + \|w\|\|\Gamma\|k_0$$

where k_0 is an unknown upper bound on $\|\delta_0\|$.

The term $-k\|w\|^2\|\Gamma(t,x)\|^2 + \|w\|\|\Gamma\|k_0$ attains a maximum value $k_0^2/4k$ at $\|w\|\|\Gamma\| = k_0/2k$. Therefore

$$\dot{V} \leq -\alpha_3(\|x\|) + \frac{k_0^2}{4k}$$

Then, the solution of the closed-loop system is uniformly bounded.