Nonlinear Systems and Control

Lecture 6 (Meetings 20-22)
Chapter 12: Feedback Control
Chapter 13: Feedback Linearization

Eugenio Schuster

schuster@lehigh.edu
Mechanical Engineering and Mechanics
Lehigh University

Approximate Input-State Linearization

We consider the system

$$\dot{x} = f(x, u), \qquad f(0,0) = 0$$

Linearization (approximation):

$$\dot{x} = Ax + Bu, \quad A = \left.\frac{\partial f}{\partial x}\right|_{(0,0)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(0,0)}.$$

State feedback:

$$u = Kx$$

with

$$P(A + BK) + (A + BK)^T P = -Q$$

Closed-loop $\dot{x} = f(x, Kx)$ is *locally asymptotically stable*.

(Exact) Input-State Linearization

We consider the system

$$\dot{x} = f(x) + g(x)u \quad (1)$$
$$y = h(x) \quad (2)$$

Does there exist a change of coordinates

$$z = T(x) \quad (3)$$

and a state feedback control

$$u = \alpha(x) + \beta(x)v \quad (4)$$

that transform the nonlinear system into the linear form

$$\dot{z} = Az + Bv \quad (5)$$

where (A, B) is controllable?

Example 1: Pendulum (Single link manipulator)

$$\ddot{\theta} + b\dot{\theta} + a\sin(\theta) = cT$$

Control goal: Regulate θ around δ using torque (T) control.

Example 2:

$$\begin{aligned} \dot{x}_1 &= a\sin(x_2) \\ \dot{x}_2 &= -x_1^2 + u \end{aligned}$$

Input-State Linearization

Definition: A continuously differentiable map with a continuously differentiable inverse is known as a *diffeomorphism*.

Note: The coordinate transformation $z = T(x)$ must be a diffeomorphism!

Definition 13.1: A nonlinear system

$$\dot{x} = f(x) + g(x)u$$

where $f : D \to R^n$ and $g : D \to R^{n \times p}$ are sufficiently smooth on a domain $D \subset R^n$, is said to be *feedback linearizable* (or *input-state linearizable*) if there exists a diffeomorphism $T : D \to R^n$ such that $D_z = T(D)$ contains the origin and the change of variables $z = T(x)$ transforms the nonlinear system into the form

$$\dot{z} = Az + B\gamma(T^{-1}(z))[u - \alpha(T^{-1}(z))]$$

with (A, B) controllable and $\gamma(x)$ nonsingular for all $x \in D$.

NOTE: Take $v = \alpha(x) + \gamma^{-1}(x)v$, then $\dot{z} = Az + Bv$.

Input-State Linearization

Under what conditions is a nonlinear system input-state linearizable?

- Let us assume $p = 1$ and $\gamma(x) = 1/\beta(x)$.

$$\begin{aligned} \dot{z} &= AT(x) + B\frac{1}{\beta(x)}(u - \alpha(x)), \qquad \frac{1}{\beta(x)} = \beta(x)^{-1} \neq \beta^{-1}(x) \\ \dot{z} &= \frac{\partial T}{\partial x}(f(x) + g(x)u) \end{aligned}$$

- Then, for both equations to be identical we need

$$\begin{aligned} \frac{\partial T}{\partial x}f(x) &= AT(x) - B\frac{\alpha(x)}{\beta(x)}, \\ \frac{\partial T}{\partial x}g(x) &= B\frac{1}{\beta(x)} \end{aligned}$$

Input-State Linearization

If the linear system has to be controllable, it is necessary and sufficient condition to express the system in controllable canonical form (chain of integrators)

$$A = A_c = \begin{bmatrix} 0 & 1 & & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ & & & 0 & 1 \\ 0 & \dots & & 0 & 0 \end{bmatrix}, \quad B = B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = C_c = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

NOTE: We need to solve an ODE system to find $T(x)$.

Input-Output Linearization

Linearizing the state equation does not necessarily linearize the output equation.

Input-Output linearization is more general than Input-State linearization

Example 3:

$$\begin{aligned}\dot{x}_1 &= a\sin(x_2) \\ \dot{x}_2 &= -x_1^2 + u \\ y &= x_2\end{aligned}$$

Input-Output Linearization

Lie Derivatives:

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

Lie Derivative of h with respect to f, or derivative of h along the trajectories of the system $\dot{x} = f(x)$.

$$\begin{aligned}
L_g L_f h(x) &= \frac{\partial (L_f h)}{\partial x} g(x) \\
L_f^2 h(x) &= L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x) \\
L_f^k h(x) &= L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x) \\
L_f^0 h(x) &= h(x)
\end{aligned}$$

Input-Output Linearization

Definition 13.2: The nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

is said to have relative degree r, $1 \leq r \leq n$, in a region $D_0 \subset D$ if

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, r-1; \qquad L_g L_f^{r-1} h(x) \neq 0$$

for all $x \in D_0$.

Relative degree = # of integrators between input and output

Input-Output Linearization

Example 4:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \phi(x) + u\\ y &= x_1\end{aligned}$$

Example 5:

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2 + u\\ y &= x_1\end{aligned}$$

Input-Output Linearization

Theorem 13.1: Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u \quad (6)$$
$$y = h(x) \quad (7)$$

and suppose it has relative degree $r \leq n$ in D. If $r = n$, then for every $x_0 \in D$, a neighborhood N of x_0 exists such that the map

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

restricted to N, is a diffeomorphism on N.

Input-Output Linearization

If $r < n$, then, for every $x_0 \in D$, a neighborhood N of x_0 and smooth functions $\phi(x), \ldots, \phi_{n-r}(x)$ exist such that $\frac{\partial \phi_i}{\partial x} g(x) = 0$, for $1 \leq i \leq n-r$, for all $x \in N$ and the map

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-r}(x) \\ --- \\ h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{8}$$

restricted to N is a diffeomorphism on N.

Input-Output Linearization

The change of variables (8) transforms (6)-(7) into the *Normal Form*

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)] \\ y &= C_c\xi \end{aligned}$$

where $\xi \in R_r$, $\eta \in R^{n-r}$, (A_c, B_c, C_c) is a controllable canonical form representation of a chain of r integrators,

$$\begin{aligned} f_0(\eta, \xi) &= \left.\frac{\partial \phi}{\partial x} f(x)\right|_{x=T^{-1}(z)} \\ \gamma(x) &= L_g L_f^{r-1} h(x) \text{ and } \alpha(x) = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)} \end{aligned}$$

Input-Output Linearization

The *Normal Form* decomposes the system into an external part ξ and an internal part η. The external part is linearized by the state feedback control

$$u = \alpha(x) + \beta(x)v$$

The internal dynamics is described by (6). Setting $\xi = 0$ in that equation results in

$$\dot{\eta} = f_0(\eta, 0) \tag{9}$$

which is called the *zero dynamics*. The system is said to be *minimum phase* if (9) is asymptotically stable.

Why *zero dynamics*? This name matches nicely with the fact that for a linear system, (9) is given by $\dot{\eta} = A_0\eta$, where the eigenvalues of A_0 are the zeros of the transfer function $H(s) = C(sI - A)^{-1}B$ ($D = 0$ for strictly proper systems). This is why a linear systems is referred to as *minimum phase* when all the zeros are "stable" (negative real part).

Input-Output Linearization

Example 6: Linear system

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

Definition: The relative degree r of a linear system whose transfer function is $H(s)$ is the difference between the degree of the numerator polynomial and the degree of the denominator polinomial, i.e., is the difference between the number of poles and zeros of the system, $r = n - m$.

Lemma: The relative degree of the SISO linear system $H(s)$, with state space representation A, B, C, D, is r if and only if

$$CA^i B = 0, \quad i = 0, 1, \ldots, r-2, \qquad CA^{r-1} B \neq 0$$

Input-Output Linearization

Example 7:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \phi(x) + u \\ y &= x_2 \end{aligned}$$

Input-State Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u$$

There is **NO** prespecified output.

Question: Can we find an output w.r.t. which the system has relative degree n and can be completely linearized?

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)], \qquad \xi = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

where $T(x)$ is a diffeomorphism.

Input-State Linearization

Vector Field: Mapping $f : D \to R^n$, $f = f(x)$

Lie Bracket:

$$
\begin{aligned}
ad_f g(x) &= [f, g]_{(x)} = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \qquad \text{Vector Field} \\
ad_f^0 g(x) &= g(x) \\
ad_f^k g(x) &= [f, ad_f^{k-1} g]_{(x)}
\end{aligned}
$$

Distribution:

$$
\Delta(x) = \text{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}, \quad f_i\text{'s are vector fields}
$$

At any $x \in D$, $\Delta(x)$ is a subset of R^n. $\Delta(x)$: Collection of linear spaces associated with the different x's.

Input-State Linearization

Involutivity: A distribution $\Delta(x)$ is involutive if

$$g_1, g_2 \in \Delta \Rightarrow [g_1, g_2] \in \Delta$$

Theorem 13.2: The system

$$\dot{x} = f(x) + g(x)u$$

is *feedback linearizable* if and only if there is a domain $D_0 \subset D$ such that

- the matrix $G(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in D_0$, where n is the order of the system (controllability condition);
- the distribution $D = \text{span}\{g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\}$ is involutive in D_0.

Input-State Linearization

Example 8:

$$\begin{aligned}\dot{x}_1 &= a\sin(x_2)\\ \dot{x}_2 &= -x_1^2 + u\end{aligned}$$

Feedback Linearization

Consider a partially feedback linearizable system of the form

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)]\end{aligned}$$

where

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

$T(x)$ is a diffeomorphism on a domain $D \subset R^n$, $D_z = T(D)$ contains the origin, (A, B) is controllable, $\gamma(x)$ is nonsingular for all $x \in D$, $f_0(0,0) = 0$, and $f_0(\eta, \xi)$, $\alpha(x)$, and $\gamma(x)$ are continuously differentiable.

Feedback Linearization

The state feedback control

$$
\begin{aligned}
u &= \alpha(x) + \beta(x)v \\
v &= -K\xi,
\end{aligned}
$$

where K is designed such that $(A - BK)$ is Hurwitz, reduces the system to the "triangular" form

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) && (10) \\
\dot{\xi} &= (A_c - B_c K)\xi && (11)
\end{aligned}
$$

Lemma 13.1: The origin of (10)-(11) is *asymptotically stable* if the origin of $\dot{\eta} = f_0(\eta, 0)$ is *asymptotically stable*.
Lemma 13.2: The origin of (10)-(11) is *globally asymptotically stable* if the system $\dot{\eta} = f_0(\eta, \xi)$ is *input-to-state stable*.

Control Problems

- Given the model $\rightarrow$ control theory body
- Given the goal $\rightarrow$ control problem formulation
  - Stabilization
  - Tracking
  - Disturbance rejection or attenuation
- Uncertainties $\rightarrow$ Robust Control or Adaptive Control
- Conflicting requirements (trade-off) $\rightarrow$ Optimal Control

Control Problems

State Feedback Stabilization Problem: Given the system

$$\dot{x} = f(t, x, u)$$

we design a "static" feedback control law

$$u = \gamma(t, x)$$

such that the origin $x = 0$ is a u.a.s. equilibrium point of the closed loop system

$$\dot{x} = f(t, x, \gamma(t, x))$$

This control law is called "static" feedback because it is a memoryless function of x.

Linear Systems: Pole Placement.

Control Problems

We can design a "dynamic" feedback control law

$$u = \gamma(t, x, z)$$

where z is the solution of a dynamical system driven by x, i.e.,

$$\dot{z} = g(t, x, z)$$

such that the origin $x = 0$, $z = 0$ is a u.a.s. equilibrium point of the closed loop system.

Example: Integral Control.

Control Problems

Output Feedback Stabilization Problem: Given the system

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned}$$

we design a "static" output feedback control law

$$u = \gamma(t, y)$$

or a "dynamic" output feedback control law

$$\dot{z} = g(t, y, z), \qquad u = \gamma(t, y, z)$$

such that the origin $x = 0$ (or $x = 0$, $z = 0$) is a u.a.s. equilibrium point of the closed loop system.

Linear Systems: Observers.

Control Problems

Stabilization:

- local stability
- regional stability
- semiglobal stability
- global stability

Example 9 (Example 12.1):

$$\dot{x} = x^2 + u$$

Control Problems

Tracking Problem in the Presence of Disturbances: Given the system

$$\begin{aligned}\dot{x} &= f(t,x,u,w)\\ y &= h(t,x,u,w)\\ y_m &= h_m(t,x,u,w)\end{aligned}$$

where x is the state, u is the control, w is a disturbance input, y is the controlled output, and y_m is the measured output. We design a control law to make

$$e(t) = y(t) - r(t) \approx 0, \quad \forall t \geq t_0$$

or more realistically,

$$e(t) \to 0 \text{ as } t \to \infty$$

Control Problems

- When exogenous signal w is generated by known model, asymptotic output tracking and disturbance rejection can be achieved by including such model in the feedback controller (*internal model principle*).
- In the case of constant exogenous signals, asymptotic output tracking and disturbance rejection can be achieved by including "integral action" in the controller.
- For a general time-varying signal w, the goal is just disturbance attenuation.
- Control laws for the tracking problem are classified similarly to the stabilization problem
  - State vs. output feedback
  - Static vs. dynamic feedback
  - Local, regional, semiglobal, or global tracking

Stabilization via Linearization

We consider the system

$$\dot{x} = f(x, u)$$

Linearization (approximation):

$$\dot{x} = Ax + Bu, \quad A = \left.\frac{\partial f}{\partial x}\right|_{(0,0)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(0,0)}.$$

State feedback:

$$u = -Kx$$

with Lyapunov function $V(x) = x^T P x$, $P = P^T > 0$, and

$$P(A - BK) + (A - BK)^T P = -Q, \quad Q = Q^T > 0$$

Closed-loop $\dot{x} = f(x, -Kx)$ is *locally asymptotically stable*.

Integral Control

Given the system

$$\begin{aligned} \dot{x} &= f(x,u,w) \\ y &= h(x,w) \\ y_m &= h_m(x,w) \end{aligned}$$

where x is the state, u is the control, w is a disturbance input, y is the controlled output, and y_m is the measured output. Let r be a constant reference and set $v = [r^T \quad w^T]^T$.

We want to desing a controller to make $y(t) \to r$ as $t \to \infty$. We assume that y is measured, i.e., y is a subset of y_m. The regulation is achieve by stabilizing the system at an equilibrium $y = r$.

Integral Control

Therefore, for each v we assume there is (x_{ss}, u_{ss}) s.t.

$$\begin{aligned} 0 &= f(x_{ss}, u_{ss}, w) \\ r &= h(x_{ss}, w) \end{aligned}$$

where x_{ss} is the desired equilibrium point and u_{ss} is the steady-state control needed to maintain the equilibrium. We integrate the regulation error $e = y - r$ (internal model),

$$\dot{\sigma} = e$$

Therefore, once we introduce integral action, the augmented system takes the following form:

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ \dot{\sigma} &= h(x, w) - r \end{aligned}$$

Integral Control

Stabilizing-controller structure depends on measured signal.

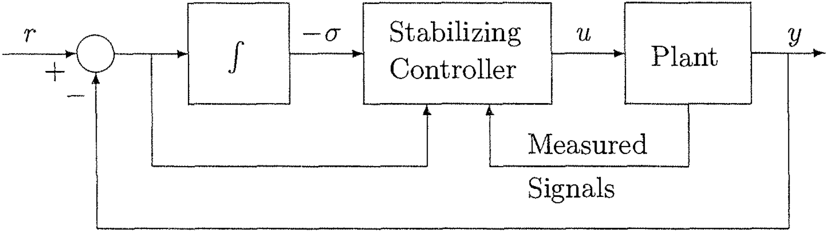

Figure: Integral control.

Integral Control

In the case of state feedback, i.e., $y_m = x$, the controller takes the form $u = \gamma(x, \sigma, e)$, where γ is designed such that there is a unique σ_{ss} that satisfies $u_{ss} = \gamma(x_{ss}, \sigma_{ss}, 0)$.

The closed loop system is in this case

$$\begin{aligned}
\dot{x} &= f(x, \gamma(x, \sigma, h(x, w) - r), w) \\
\dot{\sigma} &= h(x, w) - r \\
y &= h(x, w)
\end{aligned}$$

where the point (x_{ss}, u_{ss}) is an asymptotically stable equilibrium. At this equilibrium point, $y \equiv r$, regardless of the value of w.

Integral Control via Linearization

We propose

$$u = -K_1 x - K_2 \sigma - K_3 e$$

which results in the closed loop system

$$\begin{aligned} \dot{x} &= f(x, -K_1 x - K_2 \sigma - K_3(h(x,w) - r), w) \\ \dot{\sigma} &= h(x,w) - r \end{aligned}$$

The equilibrium point (x_{ss}, u_{ss}) satisfies

$$\begin{aligned} 0 &= f(x_{ss}, -K_1 x_{ss} - K_2 \sigma_{ss}, w) \\ 0 &= h(x_{ss}, w) - r \\ u_{ss} &= -K_1 x_{ss} - K_2 \sigma_{ss} \end{aligned}$$

Integral Control via Linearization

Linearization around (x_{ss}, u_{ss}) yields

$$\dot{\xi}_\delta = (\mathcal{A} - \mathcal{B}\mathcal{K})\xi_\delta$$

where

$$\xi_\delta = \begin{bmatrix} x - x_{ss} \\ \sigma - \sigma_{ss} \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$\begin{aligned} A &= \frac{\partial f}{\partial x}(x,u,w)\bigg|_{x=x_{ss},u=u_{ss}}, \\ B &= \frac{\partial f}{\partial u}(x,u,w)\bigg|_{x=x_{ss},u=u_{ss}}, \\ C &= \frac{\partial h}{\partial x}(x,w)\bigg|_{x=x_{ss}}, \end{aligned}$$

and $\mathcal{K}$ is designed such that $\mathcal{A} - \mathcal{B}\mathcal{K}$ is Hurwitz for all v.

Integral Control via Linearization

Example: Pendulum system

$$\ddot{\theta} = -a\sin(\theta) - b\dot{\theta} + cT$$

where $a = g/l > 0$, $b = k/m \geq 0$, $c = 1/ml^2 > 0$, θ is the angle between rod and vertical axis, and T is the torque applied to the pendulum. Goal: regulate θ to δ.

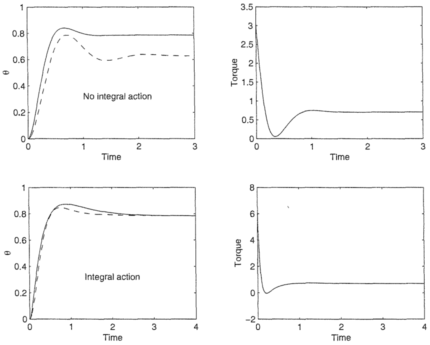

Figure: Simulation results under nominal (solid) and perturbed (dashed) parameters.

Gain Scheduling

1. Linearize the nonlinear model around a family of equilibria, parameterized by scheduling variable
2. Using linearization design a parameterized family of linear controllers to achieve specified local performance
3. Construct gain-scheduled controller such that
   - For each cte value of exogenous variable, c.l. system under gain-scheduled controller and c.l. system under fixed-gain controller have same equilibrium
   - Linearization of c.l. system under gain-scheduled controller is equivalent to linearization of the c.l. system under fixed-gain controller
4. Check the nonlocal performance of the gain-schedule controller by simulating the nonlinear closed-loop model