Nonlinear Systems and Control

Lecture 4 (Meetings 11-16)

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Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \to R^n$ is a *locally Lipschitz* map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} = 0 \in D$ is an equilibrium point of (1).

Our goal is to characterize and study stability of the equilibrium $\bar{x} = 0$ (no loss of generality).

Stability

Definition 4.1: The equilibrium point $x = 0$ of (1) is

- *stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0$$

- *unstable* if not stable
- *asymptotically stable* if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

The $\epsilon - \delta$ requirement for stability takes a challenge-answer form.

Stability

"Stability is a property of the equilibrium, not of the system"

Stability of the equilibrium is equivalent to stability of the system only when there exists only one equilibrium (e.g., linear systems). In this case stability $\equiv$ global stability.

The equilibrium point $x = 0$ of (1) is

- *attractive* if there is $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

  Example: Attractive but unstable
- *asymptotically stable (a.s.)* if it is stable and attractive.
- *globally asymptotically stable (g.a.s.)* if a.s. $\forall x(0) \in \mathcal{R}^n$.

Derivative along the trajectory

Definition: Let $V : D \to R$ be a *continuously differentiable* function defined in a domain $D \in R^n$ that contains the origin. The derivative of V along the trajectory (solution) of (1), denoted by $\dot{V}(x)$ is given by

$$\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i \\
&= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= [\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}][f_1(x), f_2(x), \ldots, f_n(x)]^T \\
&= \frac{\partial V}{\partial x} f(x)
\end{aligned}$$

Lyapunov Stability Theorem

Theorem 4.1: Let $x = 0$ be an equilibrium for (1) and $D \in R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{2}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{3}$$

Then, $x = 0$ is *stable*. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{4}$$

then $x = 0$ is *asymptotically stable*.

Lyapunov Stability Theorem

Proof:

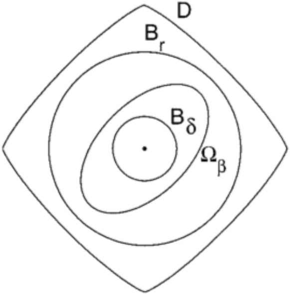

Figure: Geometric representation of sets.

- Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in R^n | \|x\| \leq r\} \subset D$$

Lyapunov Stability Theorem

- Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ by definition (2). Take $\beta \in (0, \alpha)$, and let
$$\Omega_\beta = \{x \in B_r | V(x) \leq \beta\}$$
Then, Ω_β is in the interior of B_r.
- Since Ω_β is a compact set, we conclude from Theorem 3.3 that (1) has a unique solution defined for all $t > 0$ whenever $x(0) \in \Omega_\beta$.
- Any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all time. This follows from (3) since
$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \forall t \geq 0$$
- Since $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that
$$\|x\| \leq \delta \Rightarrow V(x) < \beta$$
Then,
$$B_\delta \subset \Omega_\beta \subset B_r$$

- This implies that

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

  Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \quad \forall t \geq 0$$

  which shows that the equilibrium point $x = 0$ is stable.

- Now, assume that (4) holds as well. To show asymptotic stability, we need to show that $x(t) \to 0$ as $t \to 0$; that is, for every $a > 0$, there is $T > 0$ such that $\|x(t)\| < a$, for all $t > T$.
- By repetition of previous arguments, we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $V(x(t)) \to 0$.

Lyapunov Stability Theorem

- Since $V(x(t))$ is monotonically decreasing and bounded from below by zero, we have that

$$V(x(t)) \to c \geq 0 \text{ as } t \to \infty$$

- To show that $c = 0$, we use a contradiction argument. Suppose $c > 0$. By continuity of $V(x)$,there is $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leq \|x\| \leq r\}$. By (4), $-\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$.

Lyapunov Stability Theorem

- Lyapunov function candidate

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$

- Lyapunov function

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$
$$\dot{V}(x) \leq 0 \text{ in } D$$

Lyapunov Stability Theorem

- Lyapunov surface (level surface, level set)

$$\{x|V(x) = c\}$$

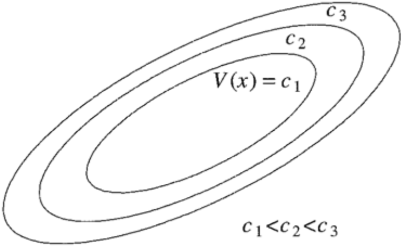

Figure: Level surfaces of a Lyapunov function.

Lyapunov Stability Theorem

- Positive definite

$$V(0) = 0, V(x) > 0, \forall x \neq 0$$

- Positive semidefinite

$$V(0) = 0, V(x) \geq 0, \forall x \neq 0$$

$V(x)$ is negative (semi)definite if $-V(x)$ is positive (semi)definite

- Lyapunov Theorem
V pdf + $\dot{V}$ nsdf $\rightarrow$ stable
V pdf + $\dot{V}$ ndf $\rightarrow$ asymptotically stable

Lyapunov Stability Theorem

Example 4.4: Consider the pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \triangleq -a\sin x_1 - bx_2\end{aligned}$$

- $V_1(x) = a(1-\cos(x_1)) + (1/2)x_2^2 \Rightarrow$ Stable.
- $V_2(x) = a(1-\cos(x_1)) + (1/2)x^T P x \Rightarrow$ Asympt. Stable.

Conclusion:

- Lyapunov's stability conditions are *only sufficient*.
- $V_1(x)$ good enough to prove a.s. via LaSalle's theorem.
- Backward approach $\rightarrow$ *Variable Gradient Method*.

Region of Attraction

When the origin $x = 0$ is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as $t \to \infty$. This gives rise to the definition of *region of attraction* (also called *region of asymptotically stability*, domain of attraction, or basin).

Definition: Let $\phi(t, x)$ be the solution of (1) that starts at initial state x at time $t = 0$. The, the region of attraction is defined as the set of all points x such that $\lim_{t \to \infty} \phi(t, x) = 0$

Question: Under what conditions will the region of attraction be the whole space R^n? In other words, for any initial state x, under what conditions the trajectory $\phi(t, x)$ approaches the origin as $t \to \infty$, no matter how large $\|x\|$ is. If an a.s. equilibrium point at the origin has this property, it is said to be *globally asymptotically stable* (g.a.s.).

Global Lyapunov Stability Theorem

Theorem 4.2: Let $x = 0$ be an equilibrium for (1). Let $V : R^n \to R$ be a continuously differentiable function, such that

$$V(0) = 0 \text{ and } V(x) > 0,\ \forall x \neq 0 \tag{5}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{6}$$

$$\dot{V}(x) < 0,\ \forall x \neq 0 \tag{7}$$

Then, $x = 0$ is *globally asymptotically stable* and is the *unique* equilibrium point.

NOTE: It is not enough to satisfy Theorem 4.1 for $D = R^n$!!!

Chetaev's Instability Theorem

Theorem 4.3: Let $x = 0$ be an equilibrium for (1). Let $V : D \to R$ be a continuously differentiable function, such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set

$$U = \{x \in B_r | V(x) > 0\}$$

where

$$B_r = \{x \in R^n | \|x\| \leq r\}.$$

Suppose that $\dot{V}(x) > 0$ in U. Then $x = 0$ is *unstable*.

Crucial Condition: $\dot{V}$ must be positive in the entire set where $V > 0$.

Proof:

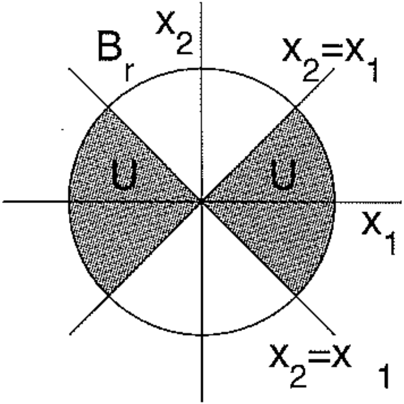

Figure: Set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2) > 0$.

Proof:

- By conditions of the theorem, the point x_0 is in the interior of U and $V(x_0) = a > 0$. The proof of this theorem is based on the fact that the trajectory $x(t)$ starting at $x(0) = x_0$ must leave the set U. To see this point, notice that as long as $x(t)$ is inside U, $V(x(t)) \geq a$ since $\dot{V}(x) > 0$ in U.
- Let

$$\gamma = \min\{\dot{V}(x) | x \in U \text{ and } V(x) \geq a\}$$

  which exists since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in U \text{ and } V(x) \geq a\} = \{x \in B_r \text{ and } V(x) \geq a\}$.
- Then, $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \geq a + \int_0^t \gamma ds = a + \gamma t$$

- This inequality shows that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) > a$. Hence, it must leave U through the sphere $\|x\| = r$. Since this can happen for an arbitrarily small $\|x_0\|$, the origin is unstable.

Example 4.7: Consider the second order system

$$\begin{aligned}\dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x)\end{aligned}$$

where $g_1()$ and $g_2()$ are locally Lipschitz functions that satisfy the inequalities

$$|g_1(x)| \leq k\|x\|^2, \qquad |g_2(x)| \leq k\|x\|^2$$

Use the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ and Chetaev's theorem to show that the origin is unstable.

Example 4.4: Consider the pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\end{aligned}$$

We consider the Lyapunov function candidate

$$V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{x_2^2}{2} \Rightarrow \dot{V}(x) = -\left(\frac{k}{m}\right)x_2^2$$

The energy Lyapunov function fails to satisfy the asymptotic stability condition of Theorem 4.1. But can $\dot{V}(x) = 0$ be maintained at $x \neq 0$?

Idea: If we can find a Lyapunov function in a domain containing the origin whose derivative along the trajectories of the system is *negative semidefinite*, and if we can establish that no trajectory can stay identicaly at points where $\dot{V}(x) = 0$ except at the origin, then the origin is asymptotically stable (LaSalle's Invariance Principle).

Invariance Principle

Let $x(t)$ be a solution of the autonomous system $\dot{x} = f(x)$.

Definition: The point p is a *positive limit point* of $x(t)$ if exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t) \to p$ as $n \to \infty$.

Definition: The set of all *positive limit points* of $x(t)$ is called the *positive limit set* of $x(t)$.

Definition: A set M is *(positively) invariant* w.r.t. $\dot{x} = f(x)$ if

$x(0) \in M \Rightarrow x(t) \in M$ for all $t \in R$ ($t \in R_+$).

Definition: $x(t)$ approaches M as $t \to \infty$ if for each $\epsilon > 0$ exists $T > 0$ such that dist$(x(t), M)(= \inf_{p \in M} \|x(t) - p\|) < \epsilon$ for all $t > T$.

$x(t) \to M$ as $t \to \infty \Rightarrow \lim_{t \to \infty}$ dist$(x(t), M) = 0$
$x(t) \to M$ as $t \to \infty$ NOT $\Rightarrow \exists \lim_{t \to \infty} x(t)$

Invariance Principle

Lemma 4.1: If $x(t)$, solution of $\dot{x} = f(x)$, is bounded for all $t \geq 0$, then it has a nonempty positive limit set L^+ which is compact and invariant. Moreover,

$$x(t) \to L^+ \text{ as } t \to \infty$$

Invariance Principle

Theorem 4.4 (LaSalle's Theorem): Let $\Omega \subset D$ be a compact set that is positively invariant w.r.t. $\dot{x} = f(x)$. Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then, every solution starting in Ω approaches M as $t \to \infty$.

Note: Unlike Lyapunov's theorem, LaSalle's theorem does NOT require the function $V(x)$ to be positive definite.

Note: When we are interested in showing that $x(t) \to 0$ as $t \to \infty$, we need to establish that the largest invariant set in E is the origin.

Invariance Principle

Proof:

- Let $x(t)$ be a solution of $\dot{x} = f(x)$ starting in Ω.
- Since $\dot{V}(x) \leq 0$ in Ω, $V(x(t))$ is a decreasing function of t.
- Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$.
- Note also that the positive limit set L^+ is in Ω because Ω is a closed set. For any $p \in L^+$, there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$.
- By continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Then, $V(x) = a$ on L^+.
- Since (by Lemma 4.1) L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+. Thus

$$L^+ \subset M \subset E \subset \Omega$$

- Since $x(t)$ is hounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 4.1). Hence, $x(t)$ approaches M as $t \to \infty$.

Invariance Principle

Corollary 4.1 (4.2): Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : D(R^n) \to R$ be a continuously differentiable (radially unbounded, positive definite) function on a domain D containing the origin (on R^n), such that $\dot{V}(x) \leq 0$ in D (in R^n). Let $S = \{x \in D(R^n) | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is *(globally) asymptotically stable*.

Invariance Principle

LaSalle's theorem:

- Relaxes the negative definiteness requirement for $\dot{V}(x)$ of Lyapunov's theorem
- Does not require $V(x)$ to be positive definite
- Gives an estimate of the region of attraction Ω which is not necessarily a level set of $V(x)$, i.e., $\Omega_c = \{x \in \mathcal{R}^n | V(x) \leq c\}$
- Applies not only to equilibrium points but also to equilibrium sets

Invariance Principle

Example:

$$\begin{aligned}\dot{x} &= -|x|x + (1-|x|)xy \\ \dot{y} &= -\frac{1}{8}(1-|x|)x^2\end{aligned}$$

Linear Systems and Linearization

The linear time-invariant system

$$\dot{x} = Ax$$

has an equilibrium point at the origin.

- if $\det(A) \neq 0$ the equilibrium point is isolated
- if $\det(A) = 0$ the system has an equilibrium subspace (nontrivial null space of A)

Note: A linear system CANNOT have multiple isolated equilibrium points.

For a given initial state $x(0)$, the solution of the system is

$$x = e^{At}x(0) \tag{8}$$

For all A, there exists a nonsingular transformation matrix P (possibly complex) that transforms A into its Jordan form J, i.e.

$$P^{-1}AP = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix},$$

where J_i is a Jordan block of order m_i associated with eigenvalue λ_i of A, i.e.,

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_i & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & \lambda_i \end{bmatrix}_{m_i \times m_i}$$

Note that a Jordan block of order $m_i = 1$ takes the form $J_i = \lambda_i$.

Then,

$$e^{At} = Pe^{Jt}P^{-1} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} e^{(\lambda_i t)} R_{ik} \tag{9}$$

where m_i is the order of the Jordan block J_i. If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i (q_i is the multiplicity of λ_i as a zero of $\det(\lambda I - A)$), then the Jordan blocks associated with λ_i have order one if and only if $\operatorname{rank}(A - \lambda_i I) = n - q_i$.

- The algebraic multiplicity of an eigenvalue λ_i of A is the number of times λ_i appears as a root of $\det(\lambda I - A)$).
- The geometric multiplicity of an eigenvalue λ_i of A is the dimension of the null space of $A - \lambda_i I$ (number of linearly independent eigenvectors).
- The algebraic multiplicity and geometric multiplicity of an eigenvalue can differ. However, the geometric multiplicity can never exceed the algebraic multiplicity.
- If for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity, then A is said to be diagonalizable.

Theorem 4.5: The equilibrium point $x = 0$ of $\dot{x} = Ax$ is

1. stable $\Leftrightarrow$ Re$\lambda_i \leq 0$ and for every eigenvalue with Re$\lambda_i = 0$ and algebraic multiplicity $q_i \geq 2$, rank$(A - \lambda_i I) = n - q_i$, where n is the dimension of x
2. (globally) asymptotically stable $\Leftrightarrow$ Re$\lambda_i < 0$ for all i

Proof:

- The origin is stable $\iff$ e^{At} in (9) is a bounded function of t for all $t > 0$.
- If Re$\lambda_i > 0$, $e^{(\lambda_i t)} \to \infty$ as $t \to \infty$. Therefore, we must restrict the eigenvalues to be in the closed left-hand complex plane.
- However, an eigenvalue on the imaginary axis (Re$\lambda_i = 0$) could give rise to unbounded terms if the order of an associated Jordan block is higher than one due to the term t^{k-1}.
- Therefore, we must restrict eigenvalues on the imaginary axis to have Jordan blocks of order one, which is equivalent to the rank condition $\mathrm{rank}(A - \lambda_i I) = n - q_i$.
- If Re$\lambda_i < 0$, $e^{(\lambda_i t)} \to 0$ as $t \to \infty$, i.e., $e^{(At)} \to 0$ as $t \to \infty$ (a.s.).
- Since $x(t)$ depends linearly on $x(0)$ (see (8)), a.s. of origin is global.

Linear Systems and Linearization

When all eigenvalues of A satisfy Re$\lambda_i < 0$, A is called a *Hurwitz matrix* or a *stability matrix*. The origin of $\dot{x} = Ax$ is a.s. if and only if A is Hurwitz.

Lyapunov function candidate

$$V(x) = x^T P x, \quad P > 0, \quad P = P^T$$

with derivative along the trajectories of $\dot{x} = Ax$

$$\dot{V}(x) = x^T(PA + A^T P)x = -x^T Q x$$

Lyapunov equation

$$PA + A^T P = -Q, \quad Q > 0, \quad Q = Q^T$$

Theorem 4.6: A matrix A is a *stability matrix* or *Hurwitz matrix*, that is, $\text{Re}(\lambda_i) < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation. Moreover, if A is a *stability matrix* or *Hurwitz matrix*, the P is the unique solution of the Lyapunov equation.

Proof:

- Sufficiency follows from Theorem 4.1 with the Lyapunov function $V(x) = x^T Px$,as we have already shown.
- To prove necessity, assume that all eigenvalues of A satisfy $\text{Re}(\lambda_i) < 0$ and consider the matrix P, defined by

$$P = \int_0^\infty e^{(A^T t)} Q e^{(At)} dt$$

- The integrand is a sum of terms of the form $t^{k-1} e^{(\lambda_i t)}$ (see (9)), where $\text{Re}(\lambda_i t) < 0$. Therefore, the integral exists. The matrix P is symmetric and positive definite. The fact that it is positive definite can be shown as follows. Supposing it is not so, there is a vector $x \neq 0$ such that $x^T Px = 0$.However,

$$\begin{aligned} x^T Px = 0 \quad &\Rightarrow \quad \int_0^\infty x^T e^{(A^T t)} Q e^{(At)} x dt = 0 \\ &\Rightarrow \quad e^{(At)} x \equiv 0, \forall t \geq 0 \Rightarrow x = 0 \end{aligned}$$

since $e^{(At)}$ is nonsingular for all t. This contradiction shows that P is positive definite.

- Now, substituting P in the Lyapunov equation yields

$$\begin{aligned} PA + A^T P &= \int_0^\infty e^{(A^T t)} Q e^{(At)} A dt + \int_0^\infty A^T e^{(A^T t)} Q e^{(At)} dt \\ &= \int_0^\infty \frac{d}{dt}\{e^{(A^T t)} Q e^{(At)}\} dt = e^{(A^T t)} Q e^{(At)}|_0^\infty = -Q \end{aligned}$$

which shows that P is indeed a solution of the Lyapunov equation.

- To show that it is the unique solution, suppose there is another solution $P \neq P$. Then,

$$(P - \bar{P})A + A^T(P - \bar{P}) = 0$$

Premultiplying by $e^{(A^T t)}$ and postmultiplying by $e^{(At)}$, we obtain

$$0 = e^{(A^T t)}[(P - \bar{P})A + A^T(P - \bar{P})]e^{(At)} = \frac{d}{dt}\{e^{(A^T t)}(P - \bar{P})e^{(At)}\}$$

Hence, $e^{(A^T t)}(P - \bar{P})e^{(At)}$ is constant $\forall t$. In particular, this is true for $t = 0$ $(e^{(At)} = I)$

$$P - \bar{P} = e^{(A^T t)}(P - \bar{P})e^{(At)} \to 0 \text{ as } t \to \infty$$

we can conclude that $P = \bar{P}$.

What if $Q = C^T C$ is positive semidefinite? The positive definiteness requirement on Q can indeed be relaxed.

Theorem 4.6 (Relaxed Conditions): A matrix A is a *stability matrix* or *Hurwitz matrix*, that is, $\text{Re}(\lambda)_i < 0$ for all eigenvalues of A, if and only if for a given positive semidefinite symmetric matrix $Q = C^T C$ there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation where the pair (A, C) is observable. Moreover, if A is *stability matrix* or *Hurwitz matrix*, the P is the unique solution of the Lyapunov equation.

Theorem 4.7 (Lyapunov's First Method or Lyapunov's Indirect Method):
Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

1. The origin is asymptotically stable if $\text{Re}\lambda_i < 0$ for all i
2. The origin is unstable if $\text{Re}\lambda_i > 0$ for some i

Note: If A has eigenvalues on imaginary axis and no eigenvalues with $\text{Re}\lambda_i > 0$ we CANNOT make conclusions.

Proof: Long but easy to follow. It uses Theorem 4.1 to prove the first statement and Theorem 4.3 to prove the second statement.

Nonautonomous Systems

Consider the nonautonomous system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \tag{10}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and *locally Lipschitz* in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point of (10) at $t = 0$ if

$$f(t, 0) = 0, \quad \forall t \geq 0$$

NOTE: The solution of (10) depends on both t and t_0. We need to refine stability definitions so that they hold uniformly in t_0.

NOTE: An equilibrium point at the origin could be a translation of a nonzero equilibrium point, or more generally, a translation of a nonzero solution (trajectory) of the system.

Nonautonomous Systems

Suppose $\bar{y}(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$. The change of variables

$$x = y - \bar{y}(\tau), \qquad t = \tau - a$$

transforms the system into the form

$$\dot{x} = g(\tau, y) - \dot{\bar{y}}(\tau) = g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \triangleq f(t, x)$$

Since

$$\dot{\bar{y}}(t + a) = g(t + a, \bar{y}(t + a)), \quad \forall t \geq 0$$

the origin $x = 0$ is an equilibrium point of the transformed system at $t = 0$.

Nonautonomous Systems

- By examining the stability behavior of the origin as an equilibrium point for the transformed system, we determine the stability behavior of the solution $\bar{y}(\tau)$ of the original system.

  Stability of trajectories $\equiv$ *Stability of equilibria of nonautonomous systems*

- If $\bar{y}(\tau)$ is not constant, the transformed system will be non-autonomous even when the original system is autonomous, i.e., even when $g(\tau, y) = g(y)$.
- The stability behavior of solutions in the sense of Lyapunov can be done only in the context of studying the stability behavior of the equilibria of non-autonomous systems.

Stability

While the solutions of autonomous systems depend only on $t - t_0$, the solutions of nonautonomous systems may depend on both t and t_0. We then refine definitions.

Definition 4.4: The equilibrium point $x = 0$ of (10) is

- *stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq t_0 \geq 0 \tag{11}$$

- *uniformly stable (u.s.)* if δ is independent of t_0
- *unstable* if not stable
- *attractive* if there is a positive constant $c = c(t_0)$ such that

$$x(t) \to 0 \text{ as } t \to \infty, \text{ for all } \|x(t_0)\| < c$$

- *uniformly attractive (u.a.)* if c is independent of t_0
- *asymptotically stable (a.s)* if stable and attractive
- *uniformly asymptotically stable (u.a.s)* if u.s. and u.a.
- *globally uniformly asymptotically stable (g.u.a.s)* if u.a.s. and c arbitrarily large

Stability

Definition 4.5: The equilibrium point $x = 0$ of (10) is *exponentially stable* if there exist positive constants c, k, and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \, \|x(t_0)\| < c \tag{12}$$

and *globally exponentially stable* if (12) is satisfied for any initial state $x(t_0)$.

Comparison Functions

The solutions of nonautonomous systems depend on both t and t_0. We refine the stability definitions, so that they hold uniformly in the initial time t_0, using special comparison functions.

Definition 4.2: A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition 4.3: A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Comparison Functions

Examples:

- $\alpha(r) = r^c \in \mathcal{K}$ if $c > 0$
- $\alpha(r) = r^c \in \mathcal{K}_\infty$ if $c > 0$
- $\beta(r, s) = r^c e^{-s} \in \mathcal{KL}$ if $c > 0$

NOTE: See more in book (Example 4.16)

Comparison Functions

The next lemma states properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions

Lemma 4.2: Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,

- α_1^{-1} is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$
- α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$
- $\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$

The next lemmas show how class $\mathcal{K}$ and class $\mathcal{KL}$ enter into Lyapunov analysis

Lemma 4.3: Let $V : D \rightarrow R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r)$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in R^n$. Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

For a quadratic positive definite function $V(x) = x^T P x$, Lemma 4.3 follows from the inequality

$$\lambda_{min}(P)\|x\|_2^2 \leq x^T P x \leq \lambda_{max}(P)\|x\|_2^2$$

Comparison Functions

Lemma 4.4: Consider the scalar autonomous differential equation

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

where α is a locally Lipschitz class $\mathcal{K}$ function defined on $[0, a)$. For all $0 \leq y_0 < a$, this equation has a unique solution $y(t)$ defined for all $t \geq t_0$. Moreover,

$$y(t) = \sigma(y_0, t - t_0)$$

where σ is a class $\mathcal{KL}$ defined on $[0, a) \times [0, \infty)$.

Lemma 4.5: The equilibrium point $x = 0$ of (10) is

- *uniformly stable (u.s.)* if and only if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \, \forall \, t \geq t_0 \geq 0, \, \forall \, \|x(t_0)\| < c \tag{13}$$

- *uniformly asymptotically stable (u.a.s)* if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \, \forall \, t \geq t_0 \geq 0, \, \forall \, \|x(t_0)\| < c \tag{14}$$

- *globally uniformly asymptotically stable (g.u.a.s.)* if and only if inequality (14) is satisfied for any initial state $x(t_0)$

Theorem 4.8: Let $x = 0$ be an equilibrium point for (10) and $D \subset R^n$ be a domain containing the origin. Let $V : [0, \infty) \times D \to R$ be a continuous differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{15}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \tag{16}$$

for all $t \geq 0$ and for all $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, $x = 0$ is *uniformly stable (u.s.)*.

Proof:

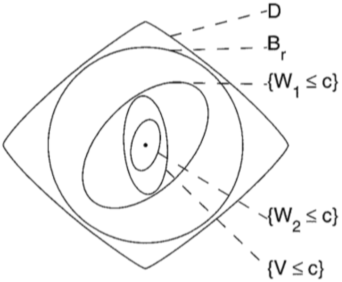

Figure: Geometric representation of sets.

- The derivative of V along the trajectories of $\dot{x} = f(x,t)$ is given by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$$

- Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$. Then, $\{x \in B_r | W_1(x) \leq c\}$ is in the interior of B_r.
- Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c} = \{x \in B_r | V(t,x) \leq c\}$$

Stability

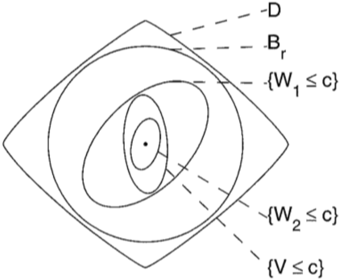

Figure: Geometric representation of sets.

- The set $\Omega_{t,c}$ contains $\{x \in B_r | W_2(x) \leq c\}$ since $W_2(x) \leq c \Rightarrow V(t,x) \leq c$. And $\Omega_{t,c}$ is a subset of $\{x \in B_r | W_1(x) \leq c\}$ since $V(t,x) \leq c \Rightarrow W_1(x) \leq c\}$.
- Thus,

$$\{x \in B_r | W_2(x) \leq c\} \subset \Omega_{t,c} \subset \{x \in B_r | W_1(x) \leq c\} \subset B_r \subset D \quad \forall t \geq 0$$

- The setup of nested sets in the figure is similar to that used in the proof of Theorem 4.1 except that surface $V(t,x) = c$ is now dependent on t, and that is why it is surrounded by time-independent surfaces $W_1(x) = c$, $W_2(x) = c$.

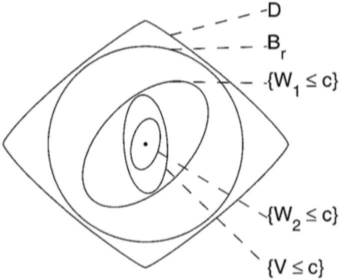

Figure: Geometric representation of sets.

- Since $\dot{V}(t,x) \leq 0$ on D, for any $t_0 > 0$ and any $x_0 \in \Omega_{t_0,c}$ the solution starting at (t_0, x_0) stays in $\Omega_{t,c}$ for all $t > t_0$.
- Therefore, any solution starting in $\{x \in B_r | W_2(x) \leq c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r | W_1(x) \leq c\}$. Hence, the solution is bounded and defined for all $t > t_0$.

Stability

- Moreover, since $\dot{V}(t,x) \leq 0$,

$$V(t, x(t)) \leq V(t_0, x(t_0)), \qquad \forall t \geq t_0$$

- By Lemma 4.3, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$ s.t.

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|)$$

- Combining the preceding two inequalities, we see that

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(V(t_0, x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$

- Since $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ (by Lemma 4.2), the inequality $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$ shows that the origin is uniformly stable (according to inequality (13) in Lemma 4.5).

Stability

Theorem 4.9: Let $x = 0$ be an equilibrium point for (10) and $D \subset R^n$ be a domain containing the origin. Let $V : [0, \infty) \times D \to R$ be a continuous differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{17}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \tag{18}$$

for all $t \geq 0$ and for all $x \in D$, where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D. Then, $x = 0$ is *uniformly asymptotically stable (u.a.s.)*.

Theorem 4.9 (cont'): Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r | W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is *globally uniformly asymptotically stable (g.u.a.s)*.

Proof: Continuation of proof of Theorem 4.8.

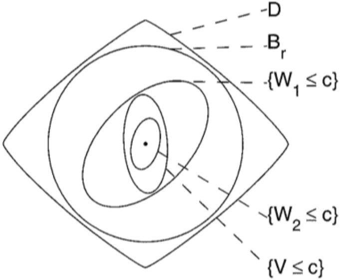

Figure: Geometric representation of sets.

- We know that trajectories starting in $\{x \in B_r | W_2(x) \leq c\}$ stays in $\{x \in B_r | W_1(x) \leq c\}$ for all $t \geq t_0$.
- By Lemma 4.3, there exist class $\mathcal{K}$ functions α_3 defined on $[0, r]$ s.t.

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \leq -\alpha_3(\|x\|)$$

Stability

- Recall that by Lemma 4.3, there exist class $\mathcal{K}$ functions α_1, α_2, defined on $[0, r]$ s.t.
$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t, x) \leq W_2(x) \leq \alpha_2(\|x\|)$$
Using the inequality
$$V \leq \alpha_2(\|x\|) \iff \alpha_2^{-1}(V) \leq \|x\| \iff \alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$$
we see that V satisfies the differential inequality
$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V)) \triangleq -\alpha(V)$$
where $\alpha \triangleq \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function defined on $[0, r]$ (see Lemma 4.2).
- Assume without loss of generality that α is locally Lipschitz. Let $y(t)$ satisfy the autonomous first-order differential equation
$$\dot{y} \leq -\alpha(y), \qquad y(t_0) = V(t_0, x(t_0)) \geq 0$$
By (the comparison) Lemma 3.4,
$$V(t, x(t)) \leq y(t), \qquad \forall t \geq t_0$$

Stability

- By Lemma 4.4, there exists a class $\mathcal{KL}$ function $\sigma(r, s)$ defined on $[0, r] \times [0, \infty)$ such that $y(t) = \sigma(y_0, t - t_0)$. Therefore,

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0), \qquad \forall V(t_0, x(t_0)) \in [0, c]$$

- Therefore, any solution starting in $\{x \in B_r | W_2(x) \leq c\}$ satisfies the inequality

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\sigma(V(t_0, x(t_0)), t - t_0)) \\ &\leq \alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|)), t - t_0)) \triangleq \beta(\|x(t_0)\|, t - t_0) \end{aligned}$$

- Lemma 4.2 shows that $\beta \triangleq \alpha_1^{-1} \circ \sigma \circ \alpha_2$ is a class $\mathcal{KL}$ function. Thus, inequality (14) in Lemma 4.5 is satisfied, which implies that $x = 0$ is uniformly asymptotically stable (u.a.s).
- if $D = R^n$, the functions α_1, α_2, and α_3 are defined on $[0, \infty)$. Hence, α and β are independent of c. As $W_1(x)$ is radially unbounded, c can be chosen arbitrarily large to include any initial state in $\{W_2(x) \leq c\}$. Thus, (14) holds for any initial state, showing that the origin globally uniformly asymptotically stable (g.u.a.s).

Theorem 4.10: Let $x = 0$ be an equilibrium point for (10) and $D \subset R^n$ be a domain containing the origin. Let $V : [0, \infty) \times D \to R$ be a continuous differentiable function such that

$$k_1 \|x\|^a \leq V(t, x) \leq k_2 \|x\|^a \tag{19}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3 \|x\|^a \tag{20}$$

for all $t \geq 0$ and for all $x \in D$, where k_1, k_2 and k_3 are positive constants. Then, $x = 0$ is *exponentially stable (e.s.)*. Moreover, if the assumptions hold globally, then $x = 0$ is *globally exponentially stable (e.s.)*.

Proof:

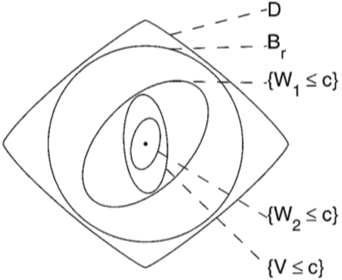

Figure: Geometric representation of sets.

- From the figure, it can be seen that trajectories starting in $\{k_2\|x\|^a \leq c\}$, for sufficiently small c, remain bounded for all $t \geq t_0$.
- The conditions of the theorem show that V satisfies the differential inequality

$$\dot{V} \leq -\frac{k_3}{k_2}V$$

Stability

- By the comparison Lemma 3.4,

$$V(t, x(t)) \leq V(t_0, x(t_0))e^{-(k_3/k_2)(t-t_0)}$$

- Hence,

$$\begin{aligned} \|x(t)\| &\leq \left[\frac{V(t, x(t))}{k_1}\right]^{1/a} \leq \left[\frac{V(t_0, x(t_0))e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} \\ &\leq \left[\frac{k_2\|x(t_0)\|^a e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} = \left(\frac{k_2}{k_1}\right)^{1/a} \|x(t_0)\|e^{-(k_3/k_2)(t-t_0)} \end{aligned}$$

  Thus, the origin is exponentially stable.

- If all the assumptions hold globally, c can be chosen arbitrarily large and the previous inequality holds for all $x(t_0) \in R^n$.

LTV Systems and Linearization

The stability behavior of the origin as an equilibrium point for the linear time varying (LTV) system

$$\dot{x} = A(t)x \tag{21}$$

can be characterized in terms of the state transition matrix of the system. The solution of (21) is given by

$$x(t) = \Phi(t, t_0)x(t_0)$$

where $\Phi(t, t_0)$ is the state transition matrix.

Theorem 4.11: The equilibrium point $x = 0$ of (21) is *(globaly) uniformly asymptotically stable* if and only if the state transition matrix satisfies the inequality

$$\|\Phi(t, t_0)\| \leq k e^{-\lambda(t-t_0)}, \qquad \forall t \geq t_0 \geq 0$$

for some positive constants k and λ.

Note: Impractical. It needs to compute $\Phi(t, t_0)$!

For linear systems with isolated equilibrium, stability results are automatically globally. Moreover,

$$\text{asymptotical stability} \iff \text{exponential stability}$$

LTV Systems and Linearization

For LTV systems, uniform asymptotic stability cannot be characterized by the location of the eigenvalues of the matrix A as it is done for LTI systems.

Example 4.22: Consider a second-order linear system with

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

For each t, the eigenvalues of $A(t)$ are given by $-0.25 \pm 0.25\sqrt{7}j$. Thus, the eigenvalues are independent of t and lie in the open left-half plane. Yet, the origin is unstable. It can be verified that

$$\Phi(t, o) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

which shows that there are initial states $x(0)$, arbitrarily close to the origin, for which the solution is unbounded and escapes to infinity.

Theorem 4.12: Let $x = 0$ be the *exponentially stable* equilibrium point of (21). Suppose $A(t)$ is continuous and bounded. Let $Q(t)$ be a continuous, bounded, positive definite, symmetric matrix. Then, there is a continuously differentiable, bounded, positive definite, symmetric matrix $P(t)$ that satisfies

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

Hence,

$$V(t, x) = x^T P(t)x$$

is a Lyapunov function for the system that satisfies the conditions of Theorem 4.10.

Note: It plays the role of Theorem 4.6 for LTV systems.

Proof: Take

$$P(t) = \int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t)d\tau$$

as gramian.

LTV Systems and Linearization

Theorem 4.13: Let $x = 0$ be and equilibrium point for the nonlinear system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to \mathcal{R}^n$ is continuously differentiable, $D = \{x \in R^n | \|x\|_2 < r\}$ and the Jacobian matrix $[\partial f / \partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

Note: It plays the role of Theorem 4.7 for LTV systems.

Proof: Apply Theorem 4.12.

Converse Theorems

Lyapunov Theorems:

$$V \text{ exists } \Rightarrow x = 0 \text{ u.a.s or e.s.} \tag{22}$$

Converse Lyapunov Theorems:

$$V \text{ exists } \Leftarrow x = 0 \text{ u.a.s or e.s.} \tag{23}$$

Note: They just guarantee existence of Lyapunov function V but do NOT tell us how to obtain that Lyapunov function!

The idea of constructing a converse Lyapunov function is not new. It has been done for linear systems in the proof of Theorem 4.12. A careful reading of that proof shows that linearity of the system does not play a crucial role in the proof, except for showing that $V(t, x)$ is quadratic in x. This observation leads to the first of our two converse theorems, whose proof is a simple extension of the proof of Theorem 4.12.

Theorem 4.14: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x) \tag{24}$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n | \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let k, λ, and r_0 be positive constants with $r_0 < r/k$. Let $D_0 = \{x \in R^n | \|x\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \, \|x(t_0)\| \in D_0, \quad \forall t \geq t_0 \geq 0 \tag{25}$$

Note: This is the definition of exponential stability!

Converse Theorems

Then, there is a function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 \tag{26}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2 \tag{27}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\| \tag{28}$$

for some positive constants c_1, c_2, c_3, and c_4. Moreover, if $r = \infty$ and the origin is globally exponentially stable, then $V(t,x)$ is defined and satisfies the aforementioned inequalities on R^n. Furthermore, if the system is autonomous, V can be chosen independent of t.

Converse Theorems

In Theorem 4.13, it was shown that if the linearization of a nonlinear system about the origin has an exponentially stable equilibrium, then the origin is an exponentially stable equilibrium for the nonlinear system. It is possible to use Theorem 4.14 to prove that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin.

Theorem 4.15: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x) \tag{29}$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n | \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0} \tag{30}$$

Then, $x = 0$ is an *exponentially stable* equilibrium point for the nonlinear system if and only if it is an *exponentially stable* equilibrium point for the linear system

$$\dot{x} = A(t)x \tag{31}$$

Converse Theorems

Corollary 4.3: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x) \tag{32}$$

where $f(x)$ is continuously differentiable in some neighborhood of the origin. Let

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=0} \tag{33}$$

Then, $x = 0$ is an *exponentially stable* equilibrium point for the nonlinear system if and only if A is Hurwitz.

Example 4.23: Consider the first-order system $\dot{x} = -x^3$. It was shown that the orign is asymptotically stable, but linearization about the origin results in the linear system $\dot{x} = 0$ whose A matrix is not Hurwitz. Using Corolary 4.3, we conclude that the origin is not exponentially stable.

Converse Theorems

The following converse Lyapunov theorem extend Theorem 4.15 but its proof is more involved. Theorem 4.16 applies to the more general case of uniformly asymptotically stable equilibria.

Theorem 4.16: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x) \tag{34}$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n | \|x\| < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded on D, uniformly in t. Let β be a class $\mathcal{KL}$ function and r_0 a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in R^n | \|x\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \, \forall \, t \geq t_0 \geq 0, \, \forall \, \|x(t_0)\| < c \tag{35}$$

Note: This is the definition of uniform asymptotic stability!

Converse Theorems

Then, there is a function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|) \tag{36}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -\alpha_3(\|x\|) \tag{37}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le \alpha_4(\|x\|) \tag{38}$$

for some positive constants α_1, α_2, α_3, and α_4 are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t.

Boundedness

Lyapunov analysis can be used to show boundedness of the solution of the state equation, even when there is no equilibrium point at the origin.

Example: $\dot{x} = -x + \delta \sin(t)$, $x(t_0) = a$, $a > \delta > 0$

The solution of this linear scalar differential equation is given by

$$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau d\tau,$$

which satisfies the bound $(|\sin \tau| \leq 1)$

$$\begin{aligned} |x(t)| &\leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau \\ &= e^{-(t-t_0)}a + \delta \left[1 - e^{-(t-t_0)}\right] \\ &= e^{-(t-t_0)}(a - \delta) + \delta \\ &\leq a \qquad \forall t \geq t_0, \quad a - \delta > 0 \end{aligned}$$

In this case the solution is said to be *uniformly bounded* and a is called the bound.

Boundedness

This a conservative bound as the time progresses because it does not take into account the exponentially decaying term. If we pick b s.t. $\delta < b < a$ we can show that

$$|x(t)| \leq b \qquad \forall t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right), \quad a-\delta > 0, b-\delta > 0$$

Note that

$$\begin{aligned}
t &\geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right) \\
-(t-t_0) &\leq -\ln\left(\frac{a-\delta}{b-\delta}\right) \\
e^{-(t-t_0)} &\leq \left(\frac{a-\delta}{b-\delta}\right)^{-1} \\
(a-\delta)e^{-(t-t_0)} &\leq b-\delta \\
(a-\delta)e^{-(t-t_0)} + \delta &\leq b
\end{aligned}$$

In this case the solution is said to be *uniformly <u>ultimately</u> bounded* and b is called the ultimate bound.

Boundedness

Showing that the solution of $\dot{x} = -x + \delta \sin(t)$, $x(t_0) = a$, $a > \delta > 0$ has the uniform boundedness and uniform ultimate boundedness properties can be done via Lyapunov analysis without using the explicit solution. Take $V(x) = x^2/2$, we calculate the derivate along the trajectories as

$$\dot{V} = x\dot{x} = -x^2 + x\delta \sin t \leq -x^2 + \delta |x|$$

- RHS is not negative definite because the term $\delta|x|$ dominates near the origin
- However, $\dot{V}$ is negative definite outside the set $\{|x| \leq \delta\}$
- With $c > \delta^2/2$, solutions starting in the set $\{V(x) \leq c\}$ will remain therein for all future time since $\dot{V}$ is negative on the boundary $V = c$. Note that the set $\{V(x) \leq c\}$ is equivalent to the set $\{|x| \leq \sqrt{2c}\}$, which includes the set $\{|x| \leq \delta\}$. Hence, the solutions are uniformly bounded.
- Moreover, if we pick ϵ s.t. $\delta^2/2 < \epsilon < c$, then $\dot{V}$ will be negative in the set $\{\epsilon \leq V(x) \leq c\}$, which implies that in this set V will decrease monotonically until the solution enters the set $\{V(x) \leq \epsilon\}$. From that time on, the solution cannot leave the set $\{V(x) \leq \epsilon\}$ $\dot{V}$ is negative on the boundary $V = \epsilon$. Thus, we conclude that the solution is uniformly ultimately bounded with the ultimate bound $|x| \leq \sqrt{2\epsilon}$, wich tends to $|x| \leq \delta$ as $\epsilon \to \delta^2/2$.

Boundedness

Definition 4.6: The solutions of $\dot{x} = f(t, x)$ are

- *uniformly bounded* if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0 such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \forall t \geq t_0$$

- *uniformly ultimately bounded* with ultimate bound b if there exists a positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0 such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 + T$$

- *globally uniformly bounded (or ultimately bounded)* if previous conditions hold for any arbitrarily large a.

Boundedness

The following Lyapunov-like theorem gives sufficient conditions for uniform/ultimate boundedness.

Theorem 4.18: Let $D \subset \mathcal{R}^n$ be a domain that contains the origin and $V : [0, \infty) \times D \to \mathcal{R}$ be a continuous differentiable function such that

$$\alpha_1 \left(\|x\|\right) \leq V(t, x) \leq \alpha_2 \left(\|x\|\right)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall \, \|x\| \geq \mu > 0$$

for all $t \geq 0$ and $x \in D$, where α_1, α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$.

Boundedness

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of $\dot{x} = f(t,x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \qquad \forall t_0 \leq t \leq t_0 + T \tag{39}$$

$$\|x(t)\| \leq \alpha^{-1}(\alpha_2(\mu)), \qquad \forall t_0 \geq t_0 + T \tag{40}$$

Moreover, if $D = \mathcal{R}^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then the inequalities above hold for any initial state $x(t_0)$, with no restriction on how large μ is.

Proof: The statement of this theorem reduces to that of Theorem 4.9 when $\mu = 0$.

Input-To-State Stability

Consider the system

$$\dot{x} = f(t, x, u) \tag{41}$$

where $f : [0, \infty) \times R^n \times R^m \to R^n$ is piecewise continuous in t and *locally Lipschitz* in x and u. The input $u(t)$ is a piecewise continuous function of t. Suppose the unforced system

$$\dot{x} = f(t, x, 0) \tag{42}$$

has a *globally uniformly asymptotically stable* equilibrium at the origin $x = 0$.

What can we say about the behavior of the system (41) in the presence of a bounded input $u(t)$?

Input-To-State Stability

Example: $\dot{x} = Ax + Bu$ The solution is given by

$$x(t) = e^{-A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{-A(t-\tau)}Bu(\tau)d\tau,$$

Using the bound $\|e^{-A(t-t_0)}\| \leq ke^{-\lambda(t-t_0)}$ to compute

$$\begin{aligned}
\|x(t)\| &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau \\
&\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda}|1 - e^{-\lambda(t-t_0)}| \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \\
&\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|
\end{aligned}$$

This estimate shows that the zero-input response decays to zero exponentially fast, while the zero-state response is bounded for every bounded input. In fact, the estimate shows more than a bounded-inputed-bounded-state property, it shows that bound on the zero-state response is proportional to the bound of the input.

Input-To-State Stability

For a general nonlinear system, it should NOT be surprising that these properties may not hold even when the origin of the unforced system (zero-input response) is globally uniformly asymptotically stable.

Example: $\dot{x} = -3x + (1 + 2x^2)u$

This system has a globally exponentially stable origin when $u \equiv 0$. Yet, when $x(0) = 2$ and $u(t) = 1$, the solution is given by

$$x(t) = \frac{3 - e^t}{3 - 2e^t},$$

which is unbounded, and it even has finite escape time!

Input-To-State Stability

Suppose we have a Lyapunov function $V(t,x)$ for the unforced system (42) and let us calculate $\dot{V}$ in presence of u. Due to the boundedness of u, it is possible in some cases to show that $\dot{V}$ is negative outside a ball of radius μ, where μ depends on $\sup \|u\|$. This is expected when $f(t,x,u)$ is Lipschitz, i.e.,

$$\|f(t,x,u) - f(t,x,0)\| \leq L\|u\|$$

Showing that $\dot{V}$ is negative outside a ball of radius μ enables us to apply Theorem 4.18, which states that $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ function over $[t_0, t_0 + T]$ (39) and by a class $\mathcal{K}$ function over $[t_0 + T, \infty)$ (40). Consequently,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \alpha^{-1}(\alpha_2(\mu))$$

Input-To-State Stability

Definition 4.7: The system (41) is said to be *input-to-state stable (ISS)* if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right) \tag{43}$$

Note: The origin of the unforced system (42) is *globally uniformly asymptotically stable*.

Input-To-State Stability

The following Lyapunov-like theorem gives a sufficient condition for ISS.

Theorem 4.19: Let $V : [0, \infty) \times R^n \to R$ be a continuous differentiable function such that

$$\alpha_1 (\|x\|) \leq V(t, x) \leq \alpha_2 (\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall \|x\| \geq \rho(\|u\|) > 0$$

for all $(t, x, u) \in [0, \infty) \times R^n \times R^m$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and $W_3(x)$ is a continuous positive definite functions on R^n.
Then, the system (41) is *input-to-state stable* with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.
The function V is called an *ISS Lyapunov function*.

Proof: Direct application of Theorem 4.18 with $\mu = \rho(\|u\|)$.

Input-To-State Stability

The following lemma is a direct consequence of the Converse Lyapunov theorem for g.e.s. (Theorem 4.14)

Lemma 4.6: Suppose $f(t, x, u)$ is *continuosly differentiable* and *globally Lipschitz* in (x, u), uniformly in t. If the unforced system (42) has a globally exponentially stable equilibrium point at the origin $x = 0$, then the system (41) is ISS.

Input-To-State Stability

Theorem: Converse also holds for Theorem 4.19!

Theorem: For the system

$$\dot{x} = f(x, u)$$

the following properties are equivalent

- the system is ISS,
- there exists a smooth ISS-Lyapunov function (i.e., satisfies conditions of Theorem 4.19)
- there exists a smooth positive definite radially unbounded function V and class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that the following dissipativity inequality is satisfied

$$\frac{\partial V}{\partial x} f(t, x) \leq -\rho_1 \left(\|x\|\right) + \rho_2 \left(\|u\|\right) \qquad (44)$$

Input-To-State Stability

Young's Inequality is key to apply this theorem

$$x^T y \leq \frac{\epsilon^p}{p}\|x\|^p + \frac{1}{\epsilon^q q}\|y\|^q, \quad \forall\, \epsilon > 0, \forall\, \frac{1}{p} + \frac{1}{q} = 1$$

Example (4.25): $\dot{x} = -x^3 + u$

Example (4.26): $\dot{x} = -x - 2x^3 + (1 + x^2)u^2$

Input-To-State Stability

An interesting application of input-to-state stability arises in the stability analysis of the cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2) \quad (45)$$

$$\dot{x}_2 = f_2(t, x_2) \quad (46)$$

where $f_1 : [0, \infty) \times R^{n_1} \times R^{n_2} \to R^{n_1}$ and $f_2 : [0, \infty) \times R^{n_2} \to R^{n_2}$ are piecewise continuous in t and *locally Lipschitz* in $x = [x_1^T \, x_2^T]^T$. Suppose

$$\dot{x}_1 = f_1(t, x_1, 0), \quad \dot{x}_2 = f_2(t, x_2)$$

have both g.u.a.s. equilibria at their respective origins.

Under what condition will the origin $x = 0$ of the cascade system possess the same property?

Lemma 4.7: Under the stated assumptions, if the system (45), with x_2 as input, is *input-to-state stable* and the origin of (46) is globally uniformly asymptotically stable, then the origin of the cascade system (45)-(46) is *globally uniformly asymptotically stable*.

Input-To-State Stability

Suppose that in the system

$$\dot{x}_1 = f_1(t, x_1, x_2, u) \quad (47)$$

$$\dot{x}_2 = f_2(t, x_2, u) \quad (48)$$

the x_1-system is ISS with respect to x_2 and u, and the x_2-system is ISS with respect to u. Then, the cascade system (47)-(48) is ISS with respect to u.