Nonlinear Systems and Control

Lecture 3 (Meetings 6-10)

Eugenio Schuster

schuster@lehigh.edu
Mechanical Engineering and Mechanics
Lehigh University

Fundamental Properties

To be a useful mathematical model of a physical system, the state equation

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

must have the following properties:

- Existence of solution.
- Uniqueness of solution.
- Continuous dependence of solution on initial conditions.
- Continuous dependence of solution on parameters.

Fundamental Properties

Non-uniqueness of Solution: Let us consider

$$\dot{x} = x^{\frac{1}{3}}$$

with solution given by

$$\frac{dx}{x^{\frac{1}{3}}} = dt \Rightarrow \frac{3}{2}\left[x(t)^{\frac{2}{3}} - x_0^{\frac{2}{3}}\right] = t$$

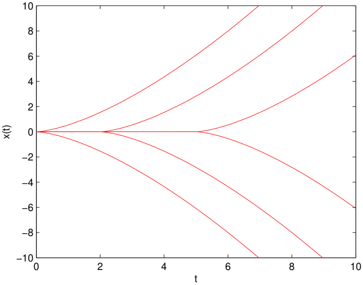


Figure: Non-uniqueness.

Fundamental Properties

If $x_0 \neq 0$, then the solution is unique

$$x(t) = sgn(x_0)\left(x_0^{\frac{2}{3}} + \frac{2}{3}t\right)^{\frac{3}{2}}$$

If $x_0 = 0$, then the solution is not unique

$$\begin{aligned} x(t) &= 0 \\ x(t) &= \pm\left(\frac{2}{3}t\right)^{\frac{3}{2}} \\ x(t) &= \begin{cases} 0 & t < T \\ \pm\left(\frac{2}{3}(t-T)\right)^{\frac{3}{2}} & t \geq T \end{cases}, \qquad \forall T \in [0, \infty) \end{aligned}$$

- Cause of non-uniqueness: non-differentiability of $x^{1/3}$ at $x = 0$.

Existence and Uniqueness of Solution

Given the state equation

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

we understand by solution a continuous function $x : [t_0, t_f] \to \mathcal{R}^n$ that satisfies the above equation $\forall t \in [t_0, t_f]$.

- The degree of freedom is $f(t, x)$.
- We have to study the conditions over $f(t, x)$ to have a unique solution.
- Continuity of $f(x, t)$ in t and x is enough for existence but not uniqueness.
    - Example: $\dot{x} = x^{1/3}$ (discussed above)
- Extra conditions must be imposed on the function $f(t, x)$.
- The Lipschitz condition is used to show existence and uniqueness.

Fundamental Properties

Lemma 3.1: Let $f : [a,b] \times D \to \mathcal{R}^m$ be continuous for some domain $D \subset \mathcal{R}^n$. Suppose that $[\partial f/\partial x]$ exists and is continuous on $[a,b] \times D$. If, for a convex subset $W \subset D$, there exists a constant $L \geq 0$ such that

$$\left\| \frac{\partial f}{\partial x} \right\| \leq L$$

on $[a,b] \times W$, then the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

is satisfied for all $t \in [a,b]$, $x \in W$ and $y \in W$. The lemma indeed shows how a Lipschitz constant can be computed using knowledge of $[\partial f/\partial x]$

Proof:

- Mean value theorem
- Hölder Inequality

Fundamental Properties

Lemma 3.2: If $f(t, x)$ and $[\partial f/\partial x]$ are continuous on $[a, b] \times D$, for some domain $D \subset \mathcal{R}^n$, then f is *locally Lipschitz* in x on $[a, b] \times D$.

$$\mathcal{C}^0 \text{ (Continuity)} \Leftarrow \text{Lipschitz} \Leftarrow \mathcal{C}^1 \text{ (Continuously Differentiability)}$$

Examples:

- Lipschitz but not $\mathcal{C}^1$: $f(x) = |x|$, $f(x) = sat(x)$.
- $\mathcal{C}^0$ but not Lipschitz: $f(x) = \sqrt{x}$.

Fundamental Properties

Lemma 3.3: If $f(t,x)$ and $[\partial f/\partial x]$ are continuous on $[a,b] \times \mathcal{R}^n$, then f is *globally Lipschitz* in x on $[a,b] \times \mathcal{R}^n$ if and only if $[\partial f/\partial x]$ is uniformly bounded on $[a,b] \times \mathcal{R}^n$.

For a globally Lipschitz function f, there exists a single Lipschitz constant L for all $x, y \in \mathcal{R}^n$. Globally Lipschitz implies $\delta = t_f - t_0$ in Theorem 3.1.

$$\mathcal{C}^1 \text{ (Continuously Differentiability)} \not\Rightarrow \text{Globally Lipschitz}$$

Example: $\mathcal{C}^1$ but not globally Lipschitz: $f(x) = x^2$.

- See also Examples 3.1, 3.2.

Fundamental Properties

Theorem 3.1 (Local Existence and Uniqueness): Let $f(t,x)$ be piecewise continuous in t and satisfy the *Lipschitz* condition

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all $x, y \in B = \{x \in \mathcal{R}^n | \|x - x_0\| \leq r\}$, for all $t \in [t_0, t_f]$ (we want the function $f(t,x)$ to be Lipschitz in a ball around the initial condition). Then, there exists $\delta > 0$ such that the state equation

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0$$

has a unique solution over $[t_0, t_0 + \delta]$.

Fundamental Properties

Proof: We start the proof by noting that if $x(t)$ is a solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{1}$$

then, by integration, $x(t)$ satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{2}$$

Conversely, if $x(t)$ satisfies (2), then $x(t)$ satisfies (1). Thus, the study of *existence and uniqueness* of the solution of the differential equation (1) is equivalent to the study of *existence and uniqueness* of the solution of the integral equation (2). This proof proceeds with the integral equation (2).

Viewing the right-hand side of (2) as a mapping of the continuous function $x : [t_0, t_1] \to R^n$, and denoting it by $(Px)(t)$, we can rewrite (2)

$$x(t) = (Px)(t) \tag{3}$$

$(Px)(t)$ is continuous in t. A solution of (3) is a *fixed point* of the mapping P that maps x into Px.

Fundamental Properties

- Existence of a fixed point of (3) can be established by the contraction mapping theorem. This requires defining a Banach space $\mathcal{X}$ and a closed set $S \subset \mathcal{X}$ such that P maps S into S and is a contraction over S.
- Let us define the Banach space as

$$\mathcal{X} = C[t_0, t_0 + \delta], \quad \text{with norm } \|x\|_C = \max_{t \in [t_0, t_0+\delta]} \|x(t)\|$$

and

$$S = \{x \in \mathcal{X} | \|x - x_0\|_C \leq r\}$$

where r is the radius of a ball B and δ is a positive constant to be chosen. Note that that $\|x(t)\|$ denotes a norm on R^n, while $\|x\|_C$ denotes a norm on $\mathcal{X}$. Also, B is a bal in R^n, while S is a ball in $\mathcal{X}$.
We restrict the choice of δ to satisfy $\delta \leq t_1 - t_0$ so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$.
- By definition, P maps $\mathcal{X}$ into $\mathcal{X}$. To show that it maps S into S, write

$$(Px)(t) - x_0 = \int_{t_0}^{t} f(s, x(s)) ds = \int_{t_0}^{t} [f(s, x(s)) - f(s, x_0) + f(s, x_0)] ds$$

Fundamental Properties

Since $f(t,x)$ is piecewise continuous, $f(t,x_0)$ is bounded on $[t_0,t_1]$. Let

$$h = \max_{t\in[t_0,t_0+\delta]} \|f(t,x)\|$$

Using the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

and the fact that for each $x \in S$,

$$\|x(t) - x_0\| \leq r, \quad \forall t \in [t_0, t_0+\delta]$$

we write

$$\begin{aligned}
\|(Px)(t) - x_0\| &\leq \int_{t_0}^{t} [\|f(s,x(s)) - f(s,x_0)\| + \|f(s,x_0)\|]ds \\
&\leq \int_{t_0}^{t} [L\|x(s) - x_0\| + h]ds \\
&\leq \int_{t_0}^{t} [Lr + h]ds = (t-t_0)(Lr+h) \leq \delta(Lr+h)
\end{aligned}$$

Hence, choosing $\delta \leq r/(Lr+h)$ ensures that P maps S into S.

Fundamental Properties

- To show that P is a contraction over S, let $x, y \in S$ and consider

$$\begin{aligned}\|(Px)(t) - (Py)(t)\| &= \left\| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))] ds \right\| \\ &\leq \int_{t_0}^{t} \|f(s, x(s)) - f(s, y(s))\| ds \\ &\leq \int_{t_0}^{t} L\|x(s) - y(s)\| ds \leq \int_{t_0}^{t} ds L\|x - y\|_C\end{aligned}$$

  Therefore

$$\|Px - Py\|_C \leq L\delta\|x - y\|_C \leq \rho\|x - y\|_C \quad \text{for } \delta \leq \frac{\rho}{L}$$

  Thus, choosing $\rho < 1$ and $\delta \leq \frac{\rho}{L}$ ensures P is a contraction mapping over S.
- By the contraction theorem, we conclude that (2) will have a unique solution if δ is chosen to satisfy

$$\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lr + h}, \frac{\rho}{L}\right\} \text{ for } \rho < 1$$

Fundamental Properties

- We are interested in establishing uniqueness in $\mathcal{X}$ and not just in S. The key is to show that any solution of (2) in $\mathcal{X}$ will lie in S. To do so, note that since $x(t_0) = x_0$ is inside the ball B, any continuous solution $x(t)$ must lie inside B for some interval of time. Suppose that $x(t)$ leaves the ball B and let $t_0 + \mu$ be the first time $x(t)$ intersects the boundary of B. Then,

$$\|x(t_0 + \mu) - x_0\| = r$$

On the other hand, for all $t \leq t_0 + \mu$,

$$\begin{aligned} \|x(t) - x_0\|| &\leq \int_{t_0}^{t} [\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|] ds \\ &\leq \int_{t_0}^{t} [L\|x(s) - x_0\| + h] ds \leq \int_{t_0}^{t} [Lr + h] ds \end{aligned}$$

Therefore,

$$r = \|x(t_0 + \mu) - x_0\| \leq [Lr + h]\mu \Rightarrow \mu \geq \frac{r}{Lr + h} \geq \delta$$

Hence, the solution $x(t)$ cannot leave the set B within the time interval $[t_0, t_0 + \delta]$, which implies that any solution in $\mathcal{X}$ lies in S. Consequently, uniqueness of the solution in S implies uniqueness in $\mathcal{X}$.

Fundamental Properties

Non-existence of Global Solution: Let us consider

$$\dot{x} = -x + x^3$$

with solution given by

$$x(t) = \frac{x_0}{\sqrt{x_0^2(1 - e^{2t}) + e^{2t}}}$$

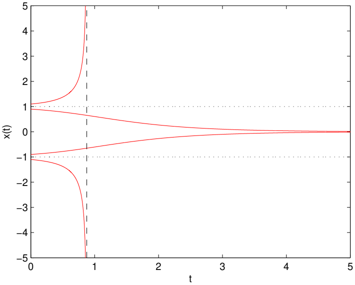


Figure: Non-existence of global solution

Fundamental Properties

If $|x_0| > 1$ then $|x(t)| \to \infty$ as $t \to \frac{1}{2} \ln \frac{x_0^2}{x_0^2 - 1}$
If $|x_0| < 1$ then $|x(t)| \to 0$ as $t \to \infty$

- Solution exists locally both in time and state.
- Cause of non-existence of global solution: $d(x^3)/dx$ not linearly bounded.

Fundamental Properties

One may try to extend the interval of existence by repeated applications of the local theorem. However, the interval of existence of the solution cannot be extended indefinitely because the condition may cease to hold.

$$\begin{bmatrix} t_0 \\ x(t_0) \end{bmatrix} \underset{\delta}{\rightarrow} \begin{bmatrix} t_0+\delta \\ x(t_0+\delta) \end{bmatrix} \underset{\delta_2}{\rightarrow} \begin{bmatrix} t_0+\delta+\delta_2 \\ x(t_0+\delta+\delta_2) \end{bmatrix} \underset{\cdots}{\rightarrow} \begin{bmatrix} T \\ x(T) \end{bmatrix}$$

$[t_0, T)$: Maximal interval of existence. In general, T may be less than t_f, in which case as $t \to T$, the solution leaves any compact set over which f is locally Lipschitz in x.

Example: $\dot{x} = x^3 \rightarrow x(t) = \frac{x_0}{\sqrt{1-2x_0^2 t}}$

Maximal interval of existence: $[0, \frac{1}{2x_0^2}]$

- See also Example 3.3.

Q: When is it guaranteed that the solution can be extended indefinitely?

Theorem 3.2 (Global Existence and Uniqueness): Let $f(t, x)$ be piecewise continuous in t and satisfy the *Lipschitz* condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y \in \mathcal{R}^n$, $t \in [t_0, t_f]$ (we want the function $f(t, x)$ to be Lipschitz globally). Then, the state equation

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

has a unique solution over $[t_0, t_f]$.

Proof: The key point of the proof is to show that the constant δ in Theorem 3.1 can be made independent of the initial state x_0. This is indeed possible thanks to the Global Lipschitz condition. So, the local result from Theorem 3.1 can indeed be used repeatedly (the solution always remain in a compact set over which f is locally Lipschitz in x).

Fundamental Properties

Q: Globally Lipschitz condition is sufficient for global existence, but is it necessary?

A: No!

Example (3.5): $\dot{x} = -x^3 = f(x)$

This function is locally but not globally Lipschitz. The Jacobian,

$$\frac{\partial f}{\partial x} = -3x^2$$

is not bounded (see Lemma 3.3). The solution

$$x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}}$$

is defined for $t \geq 0$. Unique solution!!!

In view of the conservative nature of the global Lipschitz condition, it would be useful to have a global existence and uniqueness theorem that requires the function f to be only locally Lipschitz.

Theorem 3.3: Let $f(t,x)$ be piecewise continuous in t and locally *Lipschitz* in x for all $t \geq t_0$ and all x in a domain $D \subset \mathcal{R}^n$. Let W be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$.

Proof: The proof is indeed based on the discussion on extending solutions above. By Theorem 3.1, there is a unique solution over $[t_0, t_0 + \delta]$. Let $[t_0, T]$ be its maximal interval of existence. If T is finite, the solution must leave any compact subset of D. Since by Theorem 3.3 the solution never leaves the compact set $W \subset D$, we can conclude that $T = \infty$.

Fundamental Properties

Example (3.6): $\dot{x} = -x^3 = f(x)$

We have already shown that the function is not globally Lipschitz. However, the function is locally Lipschitz in R.

- If $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative.
- If $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive.

Therefore, starting from $x(0) = a$, the solution $x(t)$ cannot leave the compact set $W = \{x \in R || x| \leq a\}$. Without calculating the solution, we conclude by Theorem 3.3 that the equation has a unique solution for all $t \geq 0$.

Continuous Dependence on IC and Parameters

Solution of

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

must depend continuously on

- initial time t_0.
- initial state x_0.
- f (parameters).

Fundamental Properties

Theorem 3.4 (Closeness of Solution): Let $f(t,x)$ be piecewise continuous in t and *Lipschitz* in x on $[t_0, t_f] \times W$ with a Lipschitz constant L, where $W \subset \mathcal{R}^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y), \qquad y(t_0) = y_0$$

and

$$\dot{z} = f(t,z) + g(t,z), \qquad z(t_0) = z_0$$

such that $y(t), z(t) \in W$ for all $t \in [t_0, t_f]$. Suppose that

$$\|g(t,x)\| \leq \mu, \quad \forall (t,x) \in [t_0, t_f] \times W$$

for some $\mu > 0$. Then, for all $t \in [t_0, t_f]$,

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \exp[L(t-t_0)] + \frac{\mu}{L}\{\exp[L(t-t_0)] - 1\}.$$

Proof: The proof is based on the direct application of the Gronwall-Bellman inequality.

Fundamental Properties

Theorem 3.5 (Continuous Dependence on IC and Parameters): Let $f(t, x, \lambda)$ be continuous in (t, x, λ) and locally *Lipschitz* in x (uniformly in t and λ) on $[t_0, t_f] \times D \times \{\|\lambda - \lambda_0\| \leq c\}$, where $D \subset \mathcal{R}^n$ in an open connected set. Let $y(t, \lambda_0)$ be a solution of

$$\dot{y} = f(t, y, \lambda_0), \qquad y(t_0, \lambda_0) = y_0 \in D.$$

Suppose $y(t, \lambda_0)$ is defined and belongs to D for all $t \in [t_0, t_f]$. Then, given $\epsilon > 0$, there is $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta \text{ and } \|\lambda - \lambda_0\| < \delta$$

then there is a unique solution $z(t, \lambda)$ of

$$\dot{z} = f(t, z, \lambda), \qquad z(t_0, \lambda) = z_0,$$

defined on $[t_0, t_f]$ and $z(t, \lambda)$ satisfies

$$\|z(t, \lambda) - y(t, \lambda_0)\| < \epsilon \quad \forall t \in [t_0, t_f].$$

Fundamental Properties

Proof: Uses Theorem 3.4 (Closeness of Solution) to show that solution stays in the tube U.

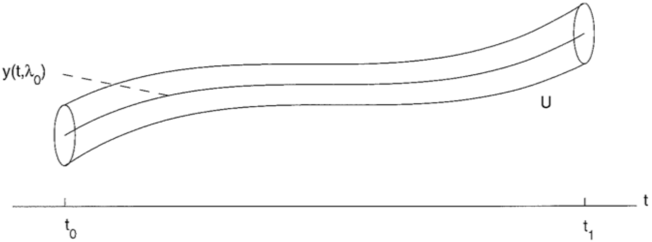


Figure: Continuous Dependence on IC and Parameters

$$U = \{(t, x) \in [t_0, t_1] \times \mathcal{R}^n | \|x - y(t, \lambda_0)\| \leq \epsilon\}$$

Sensitivity Equations

Suppose that $f(t, x, \lambda)$ is continuous in (t, x, λ) and has continuous first partial derivative with respect to x and λ for all $(t, x, \lambda) \in [t_0, t_f] \times \mathcal{R}^n \times \mathcal{R}^p$. Consider the nominal equation for $\lambda = \lambda_0$,

$$\dot{x} = f(t, x, \lambda_0), \qquad x(t_0) = x_0,$$

which has unique solution over $[t_0, t_f]$. From Theorem 3.5, for all λ sufficiently close to λ_0, i.e., $\|\lambda - \lambda_0\|$ sufficiently small, the state equation

$$\dot{x} = f(t, x, \lambda), \qquad x(t_0) = x_0$$

has a unique solution $x(t, \lambda)$ over $[t_0, t_f]$ that is close to the nominal solution $x(t, \lambda_0)$.

Fundamental Properties

We are interested in $x_\lambda(t, x, \lambda)$ (partial derivative of x w.r.t. λ). We write the integral form of the state equation,

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) ds$$

Then,

$$\begin{aligned} x_\lambda(t, \lambda) &= \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda) x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] ds \\ x_\lambda(t_0, \lambda) &= \frac{\partial x_0}{\partial \lambda} = 0 \end{aligned}$$

By defining

$$A(t, \lambda) = \left. \frac{\partial f}{\partial x}(t, x, \lambda) \right|_{x = x(t, \lambda)}, B(t, \lambda) = \left. \frac{\partial f}{\partial \lambda}(t, x, \lambda) \right|_{x = x(t, \lambda)},$$

Fundamental Properties

we can differentiate w.r.t. t and write

$$\frac{\partial}{\partial t} x_\lambda(t,\lambda) = A(t,\lambda) x_\lambda(t,\lambda) + B(t,\lambda), \quad x_\lambda(t_0,\lambda) = 0.$$

Let $S(t) = x_\lambda(t,\lambda_0)$ be the *sensitivity function*. Then, $S(t)$ is the unique solution of the *sensitivity equation*

$$\dot{S}(t) = A(t,\lambda_0) S(t) + B(t,\lambda_0), \quad S(t_0) = 0.$$

Sensitivity functions provide first-order estimates of the effect of parameter variations on solutions. They can also be used to find an approximate solution when $\|\lambda - \lambda_0\|$ is small, by Taylor expanding $x(t,\lambda)$, i.e.,

$$x(t,\lambda) \approx x(t,\lambda_0) + S(t)(\lambda - \lambda_0)$$

More on this when we study perturbation theory ...

Fundamental Properties

The procedure for calculating the sensitivity function $S(t)$ can be summarized by the following steps:

- Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$.
- Evaluate the Jacobian matrices

$$\begin{aligned} A(t,\lambda_0) &= \left.\frac{\partial f}{\partial x}(t,x,\lambda)\right|_{x=x(t,\lambda_0),\lambda=\lambda_0}, \\ B(t,\lambda_0) &= \left.\frac{\partial f}{\partial \lambda}(t,x,\lambda)\right|_{x=x(t,\lambda_0),\lambda=\lambda_0}. \end{aligned}$$

- Solve the sensitivity equation

$$\dot{S}(t) = A(t,\lambda_0)S(t) + B(t,\lambda_0), \quad S(t_0) = 0.$$

Fundamental Properties

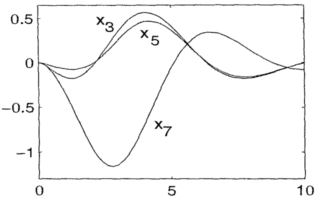
Examples:

- See Example 3.7:

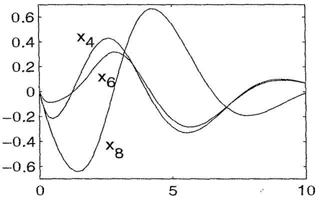
$$\begin{aligned} \dot{x}_1 &= x_2 &= f_1(x_1, x_2) \\ \dot{x}_2 &= -c \sin x_1 - (a + b \cos x_1) x_2 &= f_2(x_1, x_2) \end{aligned}$$

  – Solving the nonlinear nominal state equation and linear time-varying sensitivity equation usually demand a numerical approach.
  – These equations can be solved simultaneously

$$\begin{aligned} \dot{x} &= f(t, x, \lambda_0), & x(t_0) &= x_0 \\ \dot{S} &= \left[\frac{\partial f(t,x,\lambda)}{\partial x}\right]_{\lambda=\lambda_0} S + \left[\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_0}, & S(t_0) &= 0 \end{aligned}$$



(a)



(b)

Comparison Principle

Quite often when we study the state equation $\dot{x} = f(t,x)$ we need to compute bounds on the solution $x(t)$ without computing the solution itself.

- Gronwall-Bellman inequality.
- Comparison principle.

The upper right-hand derivative is defined as

$$D^+v(t) = \lim\sup_{h\to 0^+} \frac{v(t+h) - v(t)}{h}$$

- If $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$.
- If $\frac{1}{h}[v(t+h) - v(t)] \leq g(t,h)$ for all $h \in (0,b]$ and $\lim_{h\to 0+} g(t,h) = g_0(t)$, then $D^+v(t) \leq g_0(t)$.

Lemma 3.4 (Comparison Lemma): Consider the scalar differential equation

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0$$

where $f(t, u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and all $u \in J \subset \mathcal{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivatives $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Fundamental Properties

Examples:

- Example 3.8: $\dot{x} = -(1 + x^2)x, \qquad x(0) = a$
- Example 3.9: $\dot{x} = -(1 + x^2)x + e^t, \qquad x(0) = a$