Nonlinear Systems and Control

Lecture 2 (Meeting 5)

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Nonlinear Models

In this course we will deal with nonlinear dynamical systems that are model by a set of coupled first-order ordinary differential equations (ODE),

$$
\begin{array}{rcl}
\dot{x}_1 & = & f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 & = & f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\vdots & = & \vdots \\
\dot{x}_n & = & f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{array}
\tag{1}
$$

where $x_1, \ldots, x_n$ denote the n <u>states</u>, $u_1, \ldots, u_p$ denote the p <u>inputs</u>, t denotes time and $\dot{x}_i$ denotes the time derivative of the state x_i.

Nonlinear Models

After defining

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \qquad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

we can rewrite the *state equation* (1) as

$$\dot{x} = f(t,x,u) \tag{2}$$

which may be associated with the *output equation*

$$y = h(t,x,u) \tag{3}$$

where y denotes the q-dimensional output.

Nonlinear Models

- Nonlinear Control: Design control law

$$u = \gamma(t, x)$$

for

$$\dot{x} = f(t, x, u)$$

- Nonlinear Analysis: We study the dynamics of the unforced system

$$\dot{x} = f(t, x)$$

where u has been either forced to zero or replaced by the control law $\gamma(t, x)$.

$\dot{x} = f(t, x)$	nonautonomous or time-varying
$\dot{x} = f(x)$	autonomous or time-invariant

Nonlinear Models

A point $x = x^*$ in the state space is said to be an equilibrium point if it has the property that whenever the state of the system starts at x^*, it will remain at x^* for all future time. For the autonomous system

$$\dot{x} = f(x) \tag{4}$$

the equilibrium points are the real roots of the equation

$$0 = f(x) \tag{5}$$

Equilibrium points can be isolated or there can be a continuum of points.

Nonlinear Phenomena

How nonlinear systems are different from our well-known linear systems?

- Multiple isolated equilibria
- Finite escape time
- Limit cycles
- Chaos
- Subharmonic, harmonic or almost periodic oscillations
- Multiple modes of behavior

Nonlinear Phenomena

Multiple isolated equilibria: For linear time-invariant (LTI) systems

$$\dot{x} = Ax$$

the equilibria are given by the null space of A, $\mathcal{N}(A)$. A linear system can have only one isolated equilibrium point (A is full rank). A nonlinear system can have more than one isolated equilibrium point.

Examples:

$$\dot{x} = -x + x^3$$

The points $x = 0$ (stable) and $x = \pm 1$ (unstable) are isolated equilibrium points.

$$\dot{x} = x - x^3$$

The points $x = 0$ (unstable) and $x = \pm 1$ (stable) are isolated equilibrium points.

Nonlinear Phenomena

The solution of

$$\dot{x} = x - x^3$$

is given by

$$x(t) = \frac{x_0}{x_0^2(1 - e^{-2t}) + e^{-2t}}$$

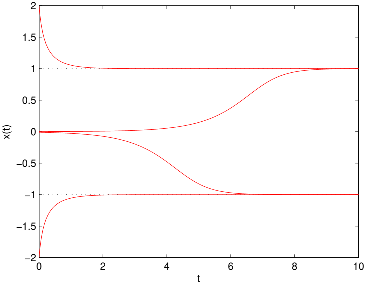


Figure: Multiple equilibria

Nonlinear Phenomena

Finite escape time: For linear time-invariant (LTI) unstable systems

$$\dot{x} = Ax$$

the state goes to infinity as time approaches infinity. For nonlinear systems, the state can go to infinity in finite time.

Examples:

$$\dot{x} = x$$

with solution

$$x(t) = x_0 e^t$$

$$\dot{x} = x^3$$

with solution

$$x(t) = \frac{x_0}{\sqrt{1 - 2x_0^2 t}}$$

Nonlinear Phenomena

Note that $x(t) = x_0 e^t$ goes to infinity when $t \to \infty$, while $x(t) = \frac{x_0}{\sqrt{1-2x_0^2 t}}$ goes to infinity when $t \to \frac{1}{2x_0^2}$.

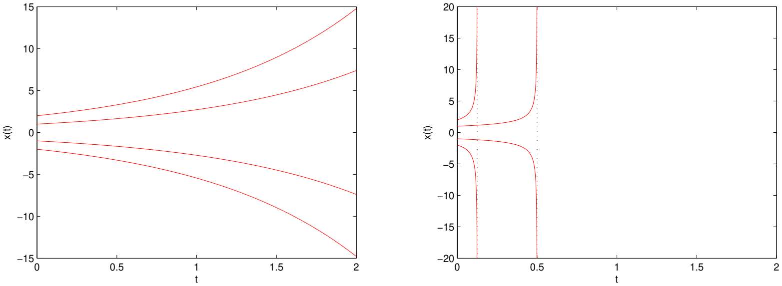


Figure: Linear system (left) with solution $x(t) = x_0 e^t$ vs. Nonlinear system (right) with solution $x(t) = \frac{x_0}{\sqrt{1-2x_0^2 t}}$.

Nonlinear Phenomena

Limit Cycles: For linear time-invariant (LTI) system to oscillate, it must have a pair of eigenvalues on the imaginary axis, which is a nonrobust condition that is almost impossible to maintain in the presence of perturbations. Even if we do, the amplitude of the oscillation will depend on the initial state.

- In real life, stable oscillators must be produced by nonlinear systems.
- There are nonlinear systems that can go into an oscillation of a fixed amplitude and frequency, irrespective of the initial state.

This type of oscillation is known as a limit cycle (isolated periodic orbit).

Nonlinear Phenomena

Examples:

Linear: $\ddot{x} + x = 0$ Nonlinear: $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$

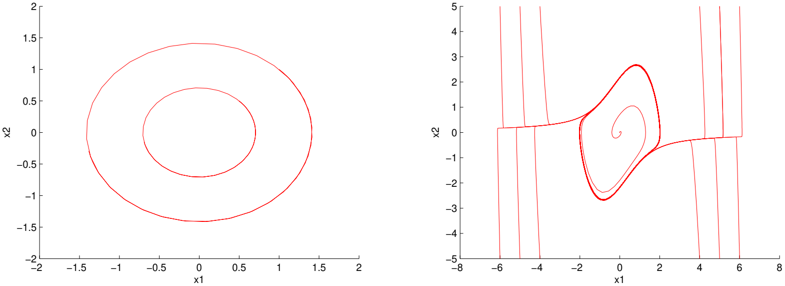


Figure: Linear system (left) vs. Nonlinear system (right)

For the limit cycle (right figure), the damping is positive for large x and negative for small x. This is a version of the well-known Van der Pol equation.

Lemma 2.1: (Poincaré-Bendixson Criterion) Consider the system

$$\dot{x} = f(x) \tag{6}$$

and let M be a closed bounded subset of the plane such that

- M contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix $[\partial f/\partial x]$ at this point has eigenvalues with positive real parts. (Hence, the equilibrium point is unstable focus or unstable node.)
- Every trajectory, starting in M stays in M for all future time.

Then, M contains a periodic orbit of (6). The lemma guarantees existence but not uniqueness.

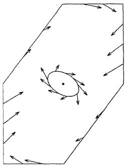


Figure: Redefinition of M to exclude unstable node or focus

Nonlinear Phenomena

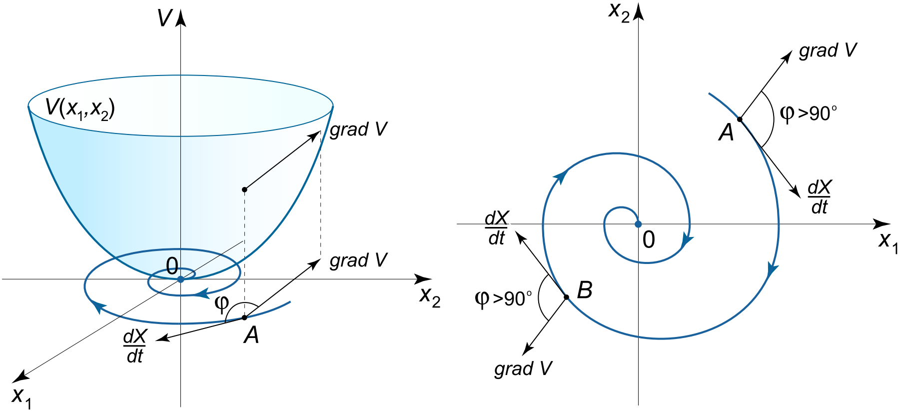
As a tool for investigating whether trajectories are trapped inside a set M, let us consider a simple closed curve defined by the equation $V(x) = c$, where $V(x)$ is continuously differentiable. The vector field $f(x)$ at a point x on the curve points inward if the inner product of $f(x)$ and the gradient vector $\nabla V(x)$ is negative, i.e.

$$f(x) \cdot \nabla V(x) = \frac{\partial V}{\partial x_1}(x) f_1(x) + \frac{\partial V}{\partial x_2}(x) f_2(x) < 0. \tag{7}$$

Let us consider the following cases:

- For a set of the form $M = \{x : V(x) \leq c\}$ for some $c > 0$, trajectories are trapped inside M if $f(x) \cdot \nabla V(x) \leq 0$ on the boundary $V(x) = c$.
- For a set of the form $M = \{x : W(x) \geq c_1 \text{ and } V(x) \leq c_2\}$ for some $c_1 > 0$ and $c_2 > 0$ (annular region), trajectories are trapped inside M if $f(x) \cdot \nabla V(x) \leq 0$ on the boundary $V(x) = c_2$ and $f(x) \cdot \nabla W(x) \geq 0$ on the boundary $W(x) = c_1$.

Nonlinear Phenomena



If the derivative $\frac{dV}{dt} = \frac{\partial V}{\partial x} f(x)$ along a phase trajectory is everywhere negative, then the trajectory tends to the origin (inward).

$\dot{V} \equiv \frac{dV}{dt}$ will be negative as long as the angle ϕ between grad $V \equiv \frac{\partial V}{\partial x}$ and $\dot{x} \equiv \frac{dx}{dt} = f(x)$ is higher than 90°.

Nonlinear Phenomena

Example: Harmonic oscillator – $\ddot{x} + x = 0$

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1
\end{aligned}
$$

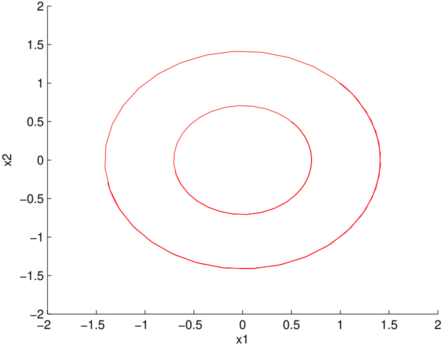


Figure: Limit cycle

Nonlinear Phenomena

Example:

$$
\begin{aligned}
\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)
\end{aligned}
$$

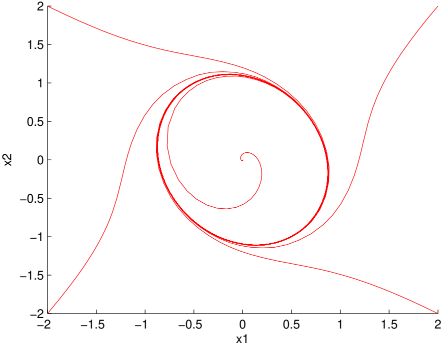


Figure: Limit cycle