Nonlinear Systems and Control

Lecture 1 (Meetings 3 & 4)

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Behavior of Second Order Systems

Consider the following linear system

$$\dot{x} = Ax \tag{1}$$

The solution of (1) for an initial condition x_0 is given by

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

where J_r is the real Jordan form of A and M is a real nonsingular matrix such that

$$J_r = M^{-1} A M$$

Behavior of Second Order Systems

There are three possible Jordan forms for a 2×2 A matrix:

- Different real eigenvalues
- Equal real eigenvalues
- Complex conjugate eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

In addition, we need to consider the case where at least one of the eigenvalues is zero.

Behavior of Second Order Systems

Different real eigenvalues: $\lambda_1 \neq \lambda_2$ (non-zero)
In this case,

$$M = [\ v_1 \quad v_2\ ]$$

where v_1 and v_2 are the real eigenvectors of A associated with λ_1 and λ_2, respectively.
The change of coordinate $z = M^{-1}x$ transforms the system into two decoupled first-order differential equations, i.e.,

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

with solution

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t} \Rightarrow z_2 = \frac{z_{20}}{(z_{10})^{\lambda_2/\lambda_1}} z_1^{\lambda_2/\lambda_1}$$

Stable Node: $\lambda_1, \lambda_2 < 0$

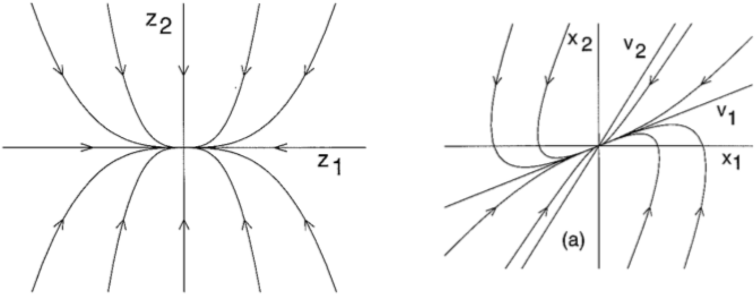

Figure: Modal (left) and original (right) coordinates.

Behavior of Second Order Systems

Unstable Node: $\lambda_1, \lambda_2 > 0$

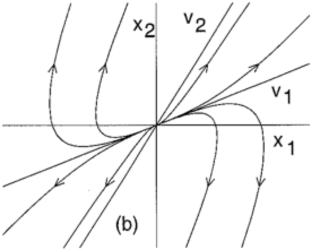

Figure: Original coordinates.

Saddle Point: $\lambda_1 > 0$, $\lambda_2 < 0$

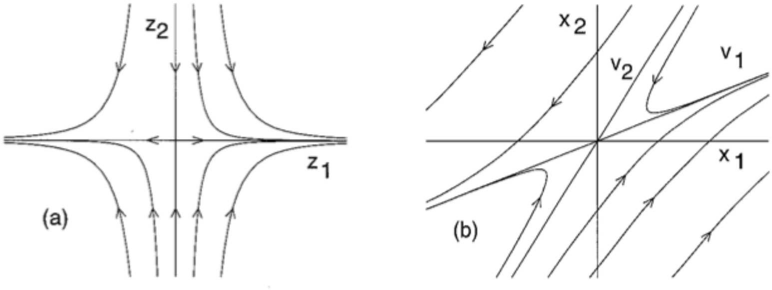

Figure: Modal (left) and original (right) coordinates.

Behavior of Second Order Systems

Complex conjugate eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$
In this case,

$$M = [\begin{array}{cc} v_1 & v_2 \end{array}]$$

where v_1 and v_2 are the real eigenvectors of A associated with λ_1 and λ_2, respectively.
The change of coordinate $z = M^{-1}x$ transforms the system into the form

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

Defining the change of coordinates

$$r = \sqrt{z_1^2 + z_2^2}, \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

Behavior of Second Order Systems

we can write the dynamic equations in polar coordinates as

$$\dot{r} = \alpha r, \qquad \dot{\theta} = \beta$$

with solution

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \theta_0 + \beta t$$

Stable Focus: $\alpha < 0$

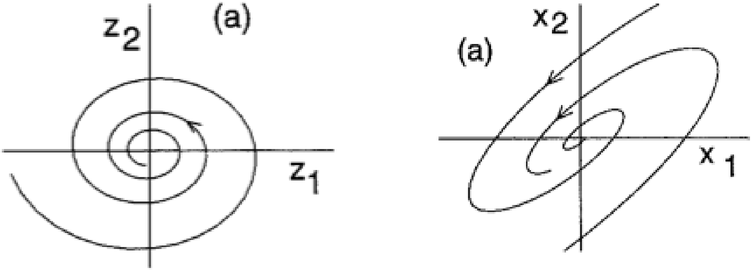

Figure: Modal (left) and original (right) coordinates.

Unstable Focus: $\alpha > 0$

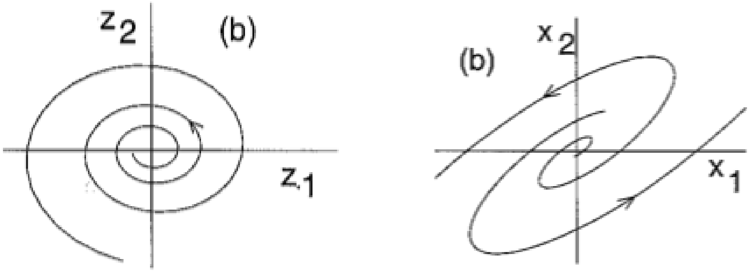

Figure: Modal (left) and original (right) coordinates.

Behavior of Second Order Systems

Center: $\alpha = 0$

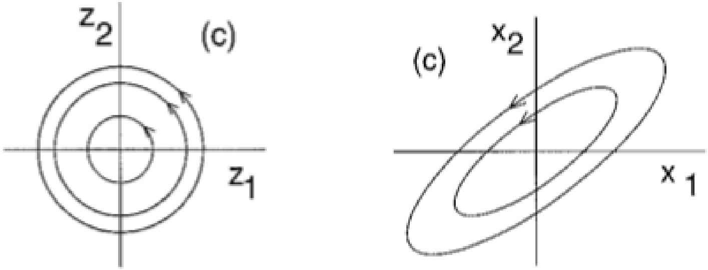

Figure: Modal (left) and original (right) coordinates.

Behavior of Second Order Systems

Equal real eigenvalues: $\lambda_1 = \lambda_2 = \lambda$ (non-zero)
In this case,

$$M = [\ v_1 \quad v_2\ ]$$

where v_1 and v_2 are the real eigenvectors of A associated with λ_1 and λ_2, respectively.
The change of coordinate $z = M^{-1}x$ transforms the system into the form

$$\dot{z}_1 = \lambda z_1 + k z_2, \qquad \dot{z}_2 = \lambda z_2$$

with solution

$$z_1(t) = (z_{10} + k z_{20} t) e^{\lambda t},\ z_2(t) = z_{20} e^{\lambda t} \Rightarrow z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln \left(\frac{z_2}{z_{20}} \right) \right]$$

Case 1: $k = 0$

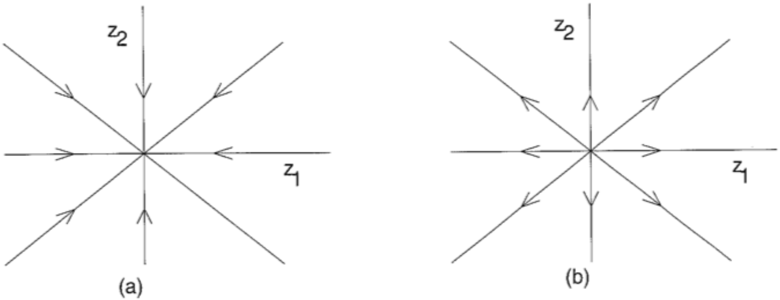

Figure: (a) $\lambda < 0$, (b) $\lambda > 0$.

Behavior of Second Order Systems

Case 2: $k = 1$

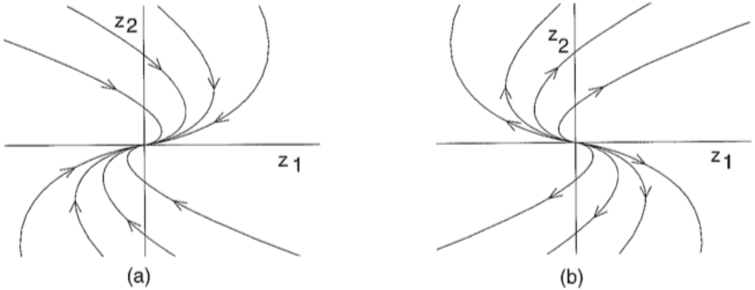

Figure: (a) $\lambda < 0$, (b) $\lambda > 0$.

Behavior of Second Order Systems

One or both zero eigenvalue: $\lambda_1 = 0, \lambda_2 \neq 0$ or $\lambda_1 = \lambda_2 = 0$
In this case A has a non-trivial null space.
When $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$M = [\ v_1 \quad v_2\ ]$$

where v_1 and v_2 are the real eigenvectors of A associated with λ_1 and λ_2, respectively. Note that v_1 spans the null space of A.
The change of coordinate $z = M^{-1}x$ transforms the system into the form

$$\dot{z}_1 = 0, \qquad \dot{z}_2 = \lambda_2 z_2$$

with solution

$$z_1(t) = z_{10}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}$$

Behavior of Second Order Systems

When $\lambda_1 = \lambda_2 = 0$,

$$M = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

where v_1 and v_2 are the real eigenvectors of A associated with λ_1 and λ_2, respectively.
The change of coordinate $z = M^{-1}x$ transforms the system into the form

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = 0$$

with solution

$$z_1(t) = z_{10} + z_{20}t, \qquad z_2(t) = z_{20}$$

Case 1: $\lambda_1 = 0$ and $\lambda_2 \neq 0$

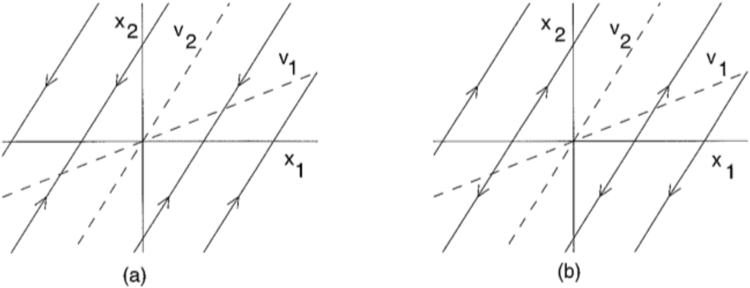

Figure: (a) $\lambda_1 = 0$, $\lambda_2 < 0$, (b) $\lambda_1 = 0$, $\lambda_2 > 0$.

Case 2: When $\lambda_1 = \lambda_2 = 0$

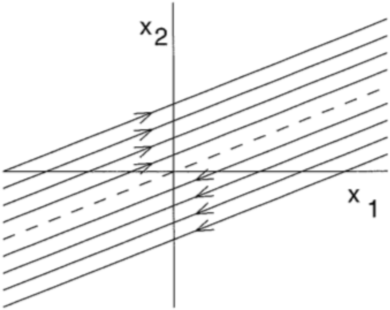

Figure: $\lambda_1 = \lambda_2 = 0$.

Qualitative Behavior Near Equilibria

Given the nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{2}$$

let us assume $p = (p_1, p_2)$ is an equilibrium point of (2), i.e.,

$$f_1(p_1, p_2) = f_2(p_1, p_2) = 0$$

Let us know expand the right-hand side of (2) around the equilibrium point p, i.e.,

$$\dot{x} = f(p) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=p} (x - p) + H.O.T.$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$$

and

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=p} = \begin{bmatrix} \frac{\partial f_1(x)}{x_1} & \frac{\partial f_1(x)}{x_2} \\ \frac{\partial f_2(x)}{x_1} & \frac{\partial f_2(x)}{x_2} \end{bmatrix}_{x=p}$$

is the Jacobian evaluated at the equilibrium point p. Since we are interested in the behavior near p, we define

$$A \equiv \left.\frac{\partial f(x)}{\partial x}\right|_{x=p}, \qquad y = x - p$$

and we obtain

Qualitative Behavior Near Equilibria

$$\dot{y} \approx Ay$$

which represents the Jacobi linearization.

Q: Is the system's linearization a good approximation of its local behavior?
A: Yes, but provided the linearization has no eigenvalue on the imaginary axis, i.e., provided the equilibrium is hyperbolic.

Therefore, as long as $f_1(x)$ and $f_2(x)$ have continuous first partial derivatives, we can conclude that

stable/unstable node		stable/unstable node
stable/unstable focus	remains	stable/unstable focus
saddle point		saddle point

Qualitative Behavior Near Equilibria

Example: hyperbolic case - Pendulum with friction

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin(\theta)$$

Qualitative Behavior Near Equilibria

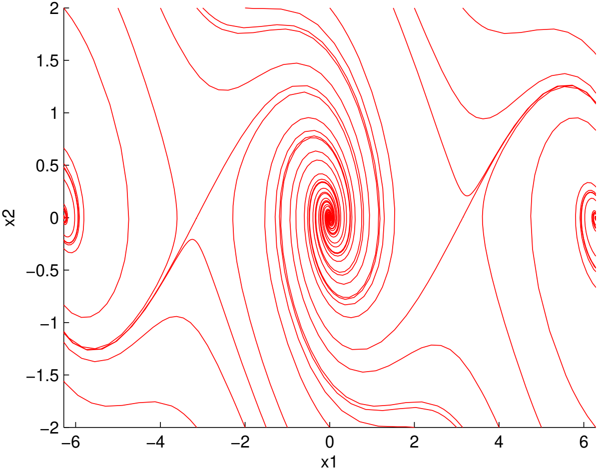

Figure: Pendulum: Saddle + Stable Focus.

Example: non-hyperbolic case

$$\begin{aligned}\dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2)\end{aligned}$$

Qualitative Behavior Near Equilibria

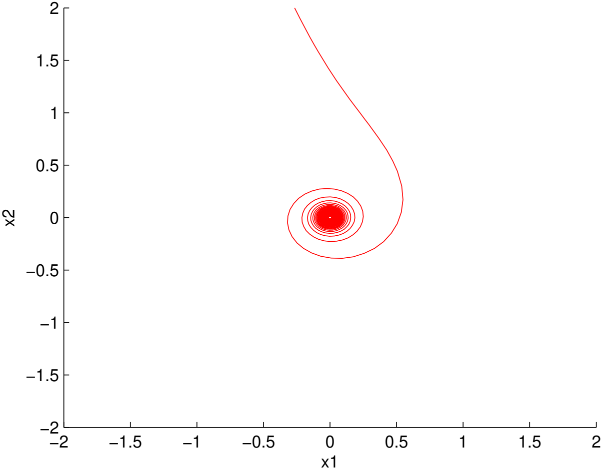

Figure: Stable/Unstable Focus.