Nonlinear Systems and Control

Lecture 1 (Meetings 1 & 2)

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What is Feedback Control for?

A feedback controller is designed for:

$$\left.\begin{array}{l} \star \text{ Stability} \\ \star \text{ Performance} \\ \quad - \text{ Equilibrium Regulation} \\ \quad - \text{ Reference Tracking} \\ \quad - \text{ Disturbance Rejection} \end{array}\right\} \text{CHANGE DYNAMIC BEHAVIOR}$$

First step in this design process: DYNAMIC MODEL

DYNAMIC MODEL $\Longrightarrow$ CONTROL DESIGN TECHNIQUE

CONTROL OBJECTIVE $\Longrightarrow$ CONTROL DESIGN TECHNIQUE

Model Classification

Model Representation ⟷ **Control Technique**

Spatial Dependence	Lumped parameter system $f = f(t)$ Ordinary Diff. Eq. (ODE)	Distributed parameter system $f = f(t, x)$ Partial Diff. Eq. (PDE)
Linearity	Linear	Nonlinear
Temporal Representation	Continuous-time	Discrete-time
Domain Representation	Time	Frequency

Model Classification: Spatial Dependence

Distributed Parameter Systems

PDE $\boxed{n_e = n_e(t,r)}$

$$\frac{\partial n_e}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r} r\left(D_n \frac{\partial n_e}{\partial r} - n_e V_n\right) + S_n(t,r)$$

$$\left.\frac{\partial n_e}{\partial r}\right|_{r=0} = 0 \qquad \left. n_e \right|_{r=a} = n_e^a$$

Lumped Parameter Systems

ODE $\boxed{n_e = n_e(t)}$

$$\frac{dn_e}{dt} = -\frac{1}{\tau_e} n_e + S_n(t)$$

$$n_e(0) = n_{e0}$$

1 Reduction

2 Keep the PDE representation (problem specific)

Control: **Interior** **Boundary**

Linearity: **Nonlinear/Linear**

$\boxed{\text{Linear/Nonlinear Distributed Parameter Control}}$

Model Classification: Linearity

Nonlinear (ODE) Systems

$$y^{(n)} = g\left(t, y^{(n-1)}, \cdots, y, u^{(m-1)}, \cdots, u\right)$$

Linear (ODE) Systems

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_{m-1} u^{(m-1)} + \cdots + b_0 u$$

Non-Autonomous

$$\dot{x} = f(t,x,u) \qquad y = h(t,x,u)$$

Autonomous

$$\dot{x} = f(x,u) \qquad y = h(x,u)$$

$$\boxed{\begin{array}{ll} x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T & \text{states} \\ u & \text{input} \\ y & \text{output} \end{array}}$$

$$\dot{x} = A(t)x + B(t)u \qquad y = C(t)x + D(t)u$$

LTV

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

LTI

1 Linearization

2 Keep the nonlinearities

$\boxed{\text{Nonlinear Control}}$

$\boxed{\text{Linear Control}}$

Output/State Feedback

Model Classification: Linearity

Nonlinear (ODE) Systems

$$y^{(n)} = g\left(t, y^{(n-1)}, \cdots, y, u^{(m-1)}, \cdots, u\right)$$

Non-Autonomous

$$\dot{x} = f(t,x,u)$$
$$y = h(t,x,u)$$

Autonomous

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

$x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ states

u input

y output

Linear (ODE) Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

LTV

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

LTI

1 Linearization

2 Keep the nonlinearities

Nonlinear Control

Linear Control

Output/State Feedback

Estimation: How to estimate states from input/output?

Model Classification: Linearity

Particular type of nonlinearities: Constraints

LTI $\quad \dot{x} = Ax + Bu \qquad u, y \rightarrow sat(u), sat(y)$ input/output constraints

$\quad y = Cx + Du \qquad \underline{x}_i < x_i < \bar{x}_i$ state constraints

$\boxed{\text{Nonlinear Optimization Techniques}}$

1 A priori $\longrightarrow$ Constraint is considered for control design

2 A posteriori $\longrightarrow$ Constraint is NOT considered for control design

$\boxed{\text{Anti-windup Techniques}}$

Model Classification: Temporal Representation

Continuous-time Systems

Discrete-time Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Sampled-Data Systems

$$y[kT] + a_{k-1}y[(k-1)T] + \cdots + a_n y[(k-n)T] = b_o u[kT] + \cdots + b_m u[(k-m)T]$$

Sampling Time

LTI
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$
LTI

Model Classification: Temporal Representation

Continuous-time Systems

Discrete-time Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Sampled-Data Systems

$$y[kT] + a_{k-1}y[(k-1)T] + \cdots + a_n y[(k-n)T] = b_o u[kT] + \cdots + b_m u[(k-m)T]$$

Sampling Time

LTI

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

LTI

System Identification: How to create models from data?

Fault Detection and Isolation: How to detect faults from data?

System Identification

Model Classification: Domain Representation

Continuous-time Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Laplace Transform

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) = \left(b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right)U(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$s = j\omega$

$$T(j\omega) = \left|T(j\omega)\right| e^{\angle T(j\omega)}$$

Discrete-time Systems

$$y[k] + a_{k-1}y[k-1] + \cdots + a_n y[k-n] = b_o u[k] + \cdots + b_m u[k-m]$$

Modern Control

Classical Control

Z Transform

$$\left(1 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^{-n}\right)Y(s) = \left(b_o + \cdots + b_m z^{-m}\right)U(s)$$

$$T(z) = \frac{Y(z)}{U(z)} = \frac{b_o + \cdots + b_m z^{-m}}{1 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^{-n}}$$

$$= K\frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

$z = e^{j\omega T}$

$$T(e^{j\omega T}) = \left|T(e^{j\omega T})\right| e^{\angle T(e^{j\omega T})}$$

T	TF
p_i	poles
z_i	zeros
K	gain

Frequency Response

Control Objectives: Optimality

Continuous-time Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

$$\updownarrow$$

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

Discrete-time Systems

$$y[k] + a_{k-1}y[k-1] + \cdots + a_n y[k-n] = b_o u[k] + \cdots + b_m u[k-m]$$

$$\updownarrow$$

$$x_{k+1} = A_k x_k + B_k u_k$$
$$y_k = C_k x_k + D_k u_k$$

$$\min_{u_k} \frac{1}{2} x_N^T S_N x_N + \frac{1}{2}\sum_{k=0}^{N-1}\left(x_k^T Q_k x_k + u_k^T R_k u_k\right)$$

$$\min_{u(t)} \frac{1}{2} x^T(T) S_T x(T) + \frac{1}{2}\int_0^T \left(x^T(t) Q(t) x(t) + u^T(t) R(t) u(t)\right) dt$$

Optimal Control

Control Objectives: Model Uncertainties

How to deal with uncertainties in the model?

A Non-model-based control

- PID
- Extremum Seeking

B Model-based control

1. Robust Control → Design for a family of plants
2. Adaptive Control → Update model (controller) in real time

$\boxed{\text{Robust \& Adaptive Control}}$

Model Classification

Model Representation ⟷ **Control Technique**

ME 433

ME 343

Spatial Dependence	Lumped parameter system $f = f(t)$ Ordinary Diff. Eq. (ODE)	Distributed parameter system $f = f(t,x)$ Partial Diff. Eq. (PDE)
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Temporal Representation	Continuous-time	Discrete-time
Domain Representation	Time	Frequency

Model Classification

Model Representation $\longleftrightarrow$ **Control Technique**

ME 433 **ME 450: NSC** **ME 343 / ME 387**

Spatial Dependence	Lumped parameter system $f = f(t)$ Ordinary Diff. Eq. (ODE)	Distributed parameter system $f = f(t, x)$ Partial Diff. Eq. (PDE)
Linearity	Linear	Nonlinear
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Controls Education in the MEM Department

ME 343: CLASSICAL CONTROL	FALL
ME 387: DIGITAL CONTROL	SPRING
ME 389: CONTROLS LAB	SPRING
ME 433: MODERN & OPTIMAL CONTROL	FALL
ME 450: ADVANCED TOPICS IN CONTROL	SPRING

NONLINEAR SYSTEMS AND CONTROL

MULTIVARIABLE ROBUST CONTROL

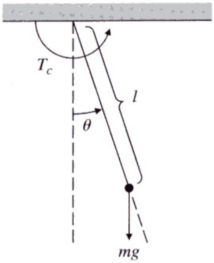
SYSTEM IDENTIFICATION

CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

ADAPTIVE CONTROL

Pendulum Control: Dynamic Model

MECHANICAL SYSTEM:



$$F = I\alpha$$ Newton's law

damping coefficient

$$I\alpha = -lmg\sin\theta - b\omega + T_c$$

$\omega = \dot{\theta}$ angular velocity

$\alpha = \dot{\omega} = \ddot{\theta}$ angular acceleration

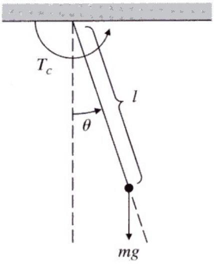
$I = ml^2$ moment of inertia

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{T_c}{ml^2}$$

Pendulum Control: Dynamic Model

MECHANICAL SYSTEM:

$$F = I\alpha$$ Newton's law



damping coefficient

$$I\alpha = -lmg\sin\theta - b\omega + T_c$$

$$\omega = \dot{\theta}$$ angular velocity

$$\alpha = \dot{\omega} = \ddot{\theta}$$ angular acceleration

$$I = ml^2$$ moment of inertia

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{T_c}{ml^2}$$

Which are the equilibrium points when T_c=0?

Stable

At equilibrium: $$\ddot{\theta} = \dot{\theta} = 0 \Rightarrow 0 = -\frac{g}{l}\sin\theta \Rightarrow \theta = 0, \pi$$

Unstable

Open loop simulations: pend_par.m, pendol01.mdl

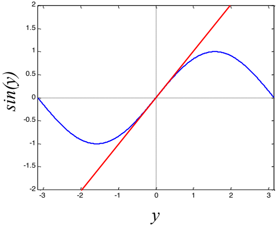
Pendulum Control: Linearization Around $\theta = 0$

What happens around $\theta=0$?

$$\theta = y \Rightarrow \qquad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}\sin(y) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

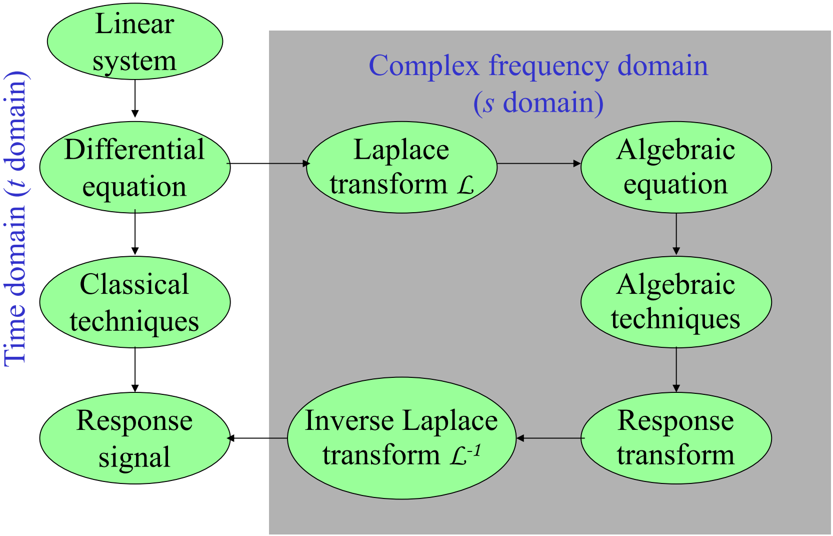
$$\sin(y) = y + h.o.t. \Rightarrow \sin(y) \approx y$$



Linearized Equation:

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$

Pendulum Control: Laplace Transform



Pendulum Control: Transfer Function Around $\theta = 0$

$$u \equiv T_c \Rightarrow \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$

$$\ddot{y} + \frac{b}{ml^2}\dot{y} + \frac{g}{l}y = \frac{u}{ml^2}$$

Laplace Transform

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s), \quad U(s) = \mathcal{L}\{u\}, \quad Y(s) = \mathcal{L}\{y\}$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml^2}s + \frac{g}{l}}$$

Characteristic Equation

Pendulum Control: Solution of the ODE

$$T_c = 0 \Rightarrow \ddot{y} + \frac{b}{ml^2}\dot{y} + \frac{g}{l}y = 0$$

What is the solutions $y(t)$?

Characteristic Equation

$$\lambda^2 + \frac{b}{ml^2}\lambda + \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 - 4\frac{g}{l}}}{2}$$

$$G(s) = \frac{1/ml^2}{s^2 + \frac{b}{ml^2}s + \frac{g}{l}}$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

The dynamics of the system is given by the roots of the denominator (poles) of the trasfer function

real(λ_1, λ_2)<0 $\Rightarrow$ STABLE SYSTEM

We use feedback control for PERFORMANCE

Pendulum Control: Linearization Around $\theta = \pi$

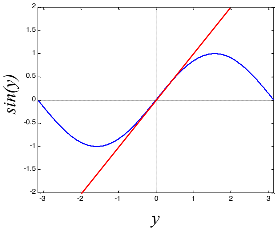
What happens around $\theta = \pi$?

$$\theta = \pi + y \Rightarrow \quad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}\sin(\pi + y) + \frac{T_c}{ml^2}$$

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} + \frac{g}{l}\sin(y) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

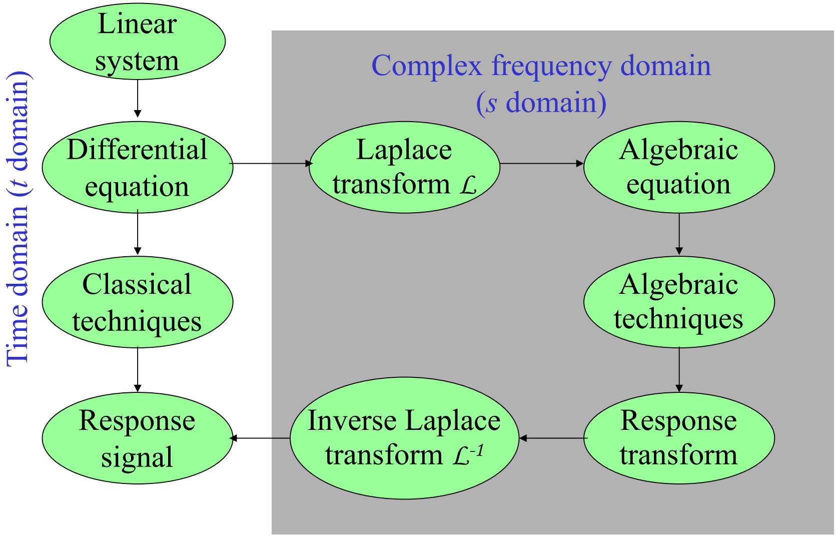
$$\sin(y) = y + h.o.t. \Rightarrow \sin(y) \approx y$$



Linearized Equation:

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} + \frac{g}{l}y + \frac{T_c}{ml^2}$$

Pendulum Control: Laplace Transform



Pendulum Control: Transfer Function Around $\theta = \pi$

$$u \equiv T_c \Rightarrow \ddot{y} = -\frac{b}{ml^2}\dot{y} + \frac{g}{l}y + \frac{T_c}{ml^2}$$

$$\ddot{y} + \frac{b}{ml^2}\dot{y} - \frac{g}{l}y = \frac{u}{ml^2}$$

Laplace Transform

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s), \quad U(s) = \mathcal{L}\{u\}, \quad Y(s) = \mathcal{L}\{y\}$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml^2}s - \frac{g}{l}}$$

Characteristic Equation

Pendulum Control: Solution of the ODE

$$T_c = 0 \Rightarrow \ddot{y} + \frac{b}{ml^2}\dot{y} - \frac{g}{l}y = 0$$

What is the solutions $y(t)$?

Characteristic Equation

$$\lambda^2 + \frac{b}{ml^2}\lambda - \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 + 4\frac{g}{l}}}{2}$$

$$G(s) = \frac{1/ml^2}{s^2 + \frac{b}{ml^2}s - \frac{g}{l}}$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

The dynamics of the system is given by the roots of the denominator (poles) of the trasfer function

real(λ_1, λ_2)>0 $\Rightarrow$ INSTABILITY

We use feedback control for STABILIZATION

Very Important Conclusion

STABILITY IS A PROPERTY OF THE EQUILIBRIUM

Pendulum Control: Design via Transfer Function

$$u \equiv T_c \Rightarrow \ddot{y} = -\frac{b}{ml^2}\dot{y} + \frac{g}{l}y + \frac{T_c}{ml^2}$$

$$\ddot{y} + \frac{b}{ml^2}\dot{y} - \frac{g}{l}y = \frac{u}{ml^2}$$

Laplace Transform

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s), \quad U(s) = \mathcal{L}\{u\}, \quad Y(s) = \mathcal{L}\{y\}$$

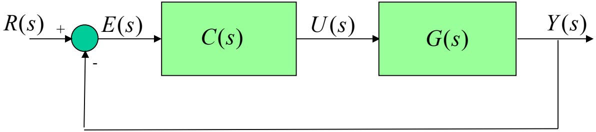
Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml^2}s - \frac{g}{l}}$$

Characteristic Equation

Pendulum Control: Design via Transfer Function

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)},$$
$$\frac{E(s)}{R(s)} = \frac{1}{1+C(s)G(s)}.$$



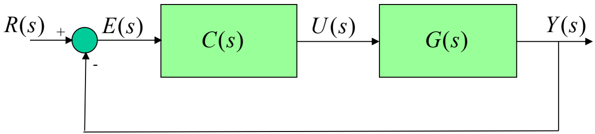
PID: Proportional – Integral – Derivative

$$C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_D s$$

$$u(t) = K_p e(t) + K_I \int_0^d e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

Closed loop simulations: pid_design.m

Pendulum Control: Design via Transfer Function



$$G(s) = \frac{Y(s)}{U(s)} = \frac{\dfrac{1}{ml^2}}{s^2 + \dfrac{b}{ml^2}s - \dfrac{g}{l}}$$

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{(1/ml^2)(K_I + K_P s + K_D s^2)}{s^3 + \dfrac{b + K_D}{ml^2}s^2 + \left(-\dfrac{g}{l} + \dfrac{K_p}{ml^2}\right)s + \dfrac{K_I}{ml^2}}$$

We can place the poles at the desired location to obtain the desired dynamics

CLASSICAL CONTROL (ME 343)

Pendulum Control: Design via State Space Model

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} + \frac{g}{l}y + \frac{T_c}{ml^2}$$

Reduce to first order equations:

State Variable Representation

$$x_1 = y \qquad x_2 = \dot{y} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{ml^2}x_2 + \frac{g}{l}x_1 + \frac{T_c}{ml^2} \end{aligned}$$

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \equiv T_c \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu$$

$$eig(A) = \left\{ \lambda : |\lambda I - A| = 0 \right\} = \left\{ \lambda : \lambda^2 + \frac{b}{ml^2}\lambda - \frac{g}{l} = 0 \right\}$$

Characteristic Equation

Pendulum Control: Design via State Space Model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu$$

$$u = -Kx = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} x$$

$$\dot{x} = (A - BK)x = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{1}{ml^2} K_1 & -\frac{b}{ml^2} - \frac{1}{ml^2} K_2 \end{bmatrix}$$

We choose K_1 and K_2 to make real(eig(A-BK))<0

MODERN CONTROL (ME 433)

Closed loop simulations: pend_par.m, statevar_control_lin.m
pendcllin01.mdl

Pendulum Control: Nonlinear State Feedback

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) - \frac{b}{ml^2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{u}{ml^2} \end{bmatrix}$$

$$u = -mgl\sin(x_1) + ml^2 v$$

Feedback Linearization

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v = A^* x + B^* v$$

$$v = -Kx = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} x \qquad \Rightarrow \dot{x} = (A^* - B^* K)x = \begin{bmatrix} 0 & 1 \\ -K_1 & -\frac{b}{ml^2} - K_2 \end{bmatrix}$$

We choose K_1 and K_2 to make real(eig(A^*-B^*K))<0

$$u = -mgl\sin(\theta - \pi) - ml^2\left[K_1(\theta - \pi) + K_2\dot{\theta}\right]$$

NONLINEAR CONTROL (ME 350/450)

Closed loop simulations: pend_par.m, statevar_control_nolin.m pendclnolin01.mdl