Multivariable Robust Control

Lecture 8 (Meetings 26-27)

Chapter 5: Performance Limitations

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Performance Limitations

Input-Output Controllability [5.1]

"Control" is not only controller design and stability analysis. Three important questions:

I. How well can the plant be controlled?

II. What control structure should be used?

What variables should we measure, which variables should we manipulate, and how are these variables best paired together?

III. How might the process be changed to improve control?

Performance Limitations

Definition

(Input-output) controllability is the ability to achieve acceptable control performance; that is, to keep the outputs (y) within specified bounds from their references (r), in spite of unknown but bounded variations, such as disturbances (d) and plant changes, using available inputs (u) and available measurements (y_m or d_m).

Note: controllability is independent of the controller, and is a property of the plant (or process) alone. It can only be affected by:

- changing the apparatus itself, e.g. type, size, etc.
- relocating sensors and actuators
- adding new equipment to dampen disturbances
- adding extra sensors
- adding extra actuators

Performance Limitations

Scaling and performance [5.1.2]:

We assume that the variables and models have been scaled so that for acceptable performance:

- Output $y(t)$ between $r-1$ and $r+1$ for any disturbance $d(t)$ between -1 and 1 and any reference $r(t)$ between $-R$ and R, using an input $u(t)$ within -1 to 1.

We interpret this definition from a frequency-by-frequency sinusoidal point of view. We then have for each frequency:

- $|e(\omega)| \leq 1$, for any disturbance $|d(\omega)| \leq 1$ and any reference $|r(\omega)| \leq R(\omega)$, using an input $|u(\omega)| \leq 1$.

Performance Limitations

Usually for simplicity:

$$\begin{aligned} R(\omega) &= R \qquad && \omega \leq \omega_r \\ R(\omega) &= 0 \qquad && \omega > \omega_r \end{aligned} \tag{5.1}$$

Because (with $r = R\widetilde{r}$):

$$e = y - r = Gu + G_d d - R\widetilde{r} \tag{5.2}$$

we can apply results for disturbances also to references by replacing G_d by $-R$.

Performance Limitations

Perfect control & plant inversion [5.2]

$$y = Gu + G_d d \tag{5.3}$$

For "perfect control", i.e. $y = r$ (not realizable) we have feedforward controller:

$$u = G^{-1}r - G^{-1}G_d d \tag{5.4}$$

With feedback control $u = K(r - y)$ we have:

$$u = KSr - KSG_d d$$

or since $T = GKS$,

$$u = G^{-1}Tr - G^{-1}TG_d d \tag{5.5}$$

Where feedback is effective ($T \approx I$) feedback input in (5.5) is the same as perfect control input in (5.4) $\Longrightarrow$ High gain feedback generates an inverse of G even though K may be very simple.

Performance Limitations

As consequence perfect control *cannot* be achieved if

- G contains RHP-zeros (since then G^{-1} is unstable)
- G contains time delay (since then G^{-1} contains a prediction)
- G has more poles than zeros (since then G^{-1} is unrealizable)

For feedforward control perfect control *cannot* be achieved if

- G is uncertain (since then G^{-1} cannot be obtained exactly)

This last restriction may be overcome by high gain feedback, but we know that we cannot have high gain feedback at all frequencies.

Because of input constraints perfect control *cannot* be achieved if

- $|G^{-1}G_d|$ is large
- $|G^{-1}R|$ is large

Constraints on S and T [5.3]

S plus T is one [5.3.1]:

$$S + T = 1 \tag{5.6}$$

- Ideally, we want S small to obtain the benefits of feedback (small control error for commands and disturbances) and T small to avoid sensitivity to noise which is one of the disadvantages of feedback.
- Unfortunately, these requirements are not simultaneously possible at any frequence as is clear from (5.6).
- Specifically, (5.6) $\Longrightarrow$ at any frequency either $|S(j\omega)| \geq 0.5$ or $|T(j\omega)| \geq 0.5$

Performance Limitations

The waterbed effects (sensitivity integrals) [5.3.2]:

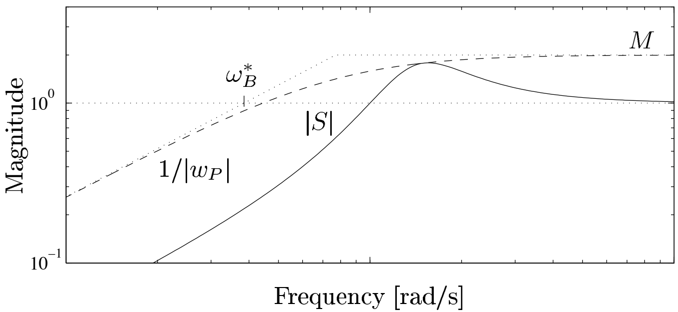


Figure 1: Plot of typical sensitivity, $|S|$, with upper bound $1/|w_P|$

Note: $|S|$ has peak greater than 1; we will show that this is unavoidable in practice. The waterbed formulae say that if we push the sensitivity down at some frequencies, then it will have to increase at others.

Pole excess of two: First waterbed formula:

Idea: When $L(s) =$ has a relative degree of two or more, then for some ω the distance between L and -1 is less than one: $|L+1| < 1 \Rightarrow |S| = |L+1|^{-1} > 1$.

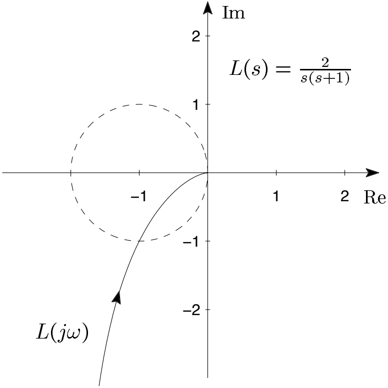


Figure 2: $|S| > 1$ whenever the Nyquist plot of L is inside the circle

Performance Limitations

Theorem

Bode Sensitivity Integral.
Suppose that the open-loop transfer function $L(s)$ *is rational and has at least two more poles than zeros (relative degree of two or more).*
Suppose also that $L(s)$ *has* N_p *RHP-poles at locations* p_i.
Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln|S(j\omega)|d\omega = \pi \cdot \sum_{i=1}^{N_p} Re(p_i) \quad (5.7)$$

where $Re(p_i)$ *denotes the real part of* p_i.

- For a stable plant we must have $\int_0^\infty \ln|S(j\omega)|d\omega = 0$. The area of sensitivity reduction ($\ln|S|$ negative) must equal the area of sensitivity increase ($\ln|S|$ positive) $\Rightarrow$ An increase in the bandwith (S smaller than 1 over a larger frequency range) must come at the expense of a larger peak in $|S|$.
- The presence of unstable poles usually increases the peak of the sensitivity.

RHP-zeros: Second waterbed formula:

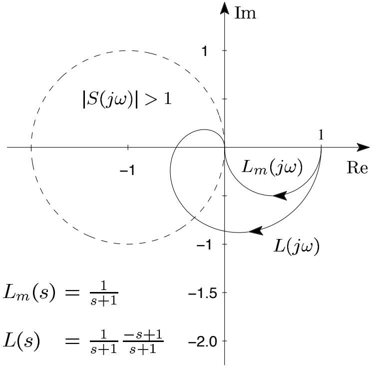


Figure 3: Additional phase lag contributed by RHP-zero causes $|S| > 1$

Performance Limitations

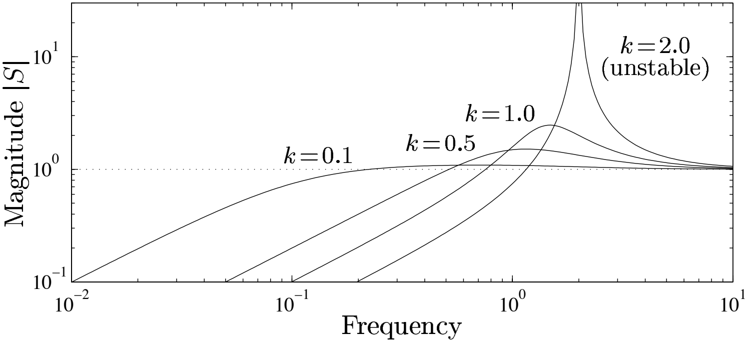


Figure 4: Effect of increased controller gain on $|S|$ for system with RHP-zero at $z = 2$, $L(s) = \frac{k}{s}\frac{2-s}{2+s}$

Performance Limitations

Theorem

Weighted sensitivity integral. *Suppose that $L(s)$ has a single real RHP-zero z and has N_p RHP-poles, p_i. Then for closed-loop stability the sensitivity function must satisfy*

$$\int_0^\infty \ln|S(j\omega)| \cdot w(z,\omega) d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{p_i - z} \right| \tag{5.8}$$

where:

$$w(z,\omega) = \frac{2z}{z^2 + \omega^2} = \frac{2}{z} \frac{1}{1 + (\omega/z)^2} \tag{5.9}$$

Performance Limitations

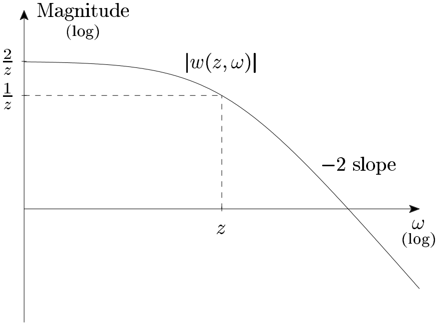


Figure 5: Plot of weight $w(z, \omega)$ for case with real zero at $s = z$

Performance Limitations

- Weight $w(z,\omega)$ "cuts off" contribution of $ln|S|$ at frequencies $\omega > z$. Thus, for a stable plant:

$$\int_0^z \ln|S(j\omega)|d\omega \approx 0 \quad (\text{ for } |S| \approx 1 \text{ at } \omega > z) \tag{5.10}$$

- The waterbed is finite, and a large peak for $|S|$ is unavoidable when we reduce $|S|$ at low frequencies (Figure 4).
- Note also that when $p_i \to z$ then $\frac{p_i+z}{p_i-z} \to \infty$. This is not surprising as such plants are in practice impossible to stabilize.

Interpolation constraints from internal stability [5.3.3]:

The two sensitivity integrals (waterbed formulas) presented above are interesting and provide valuable insights but they are less useful for a quantitative analysis of achievable performance. We are interested in determining lower bounds on S and T, which are more useful to analyze the effects of RHP zeros and RHP poles. The basis for these bounds is given by the interpolation constraints:

- If p is a RHP-pole of $L(s)$ then

$$T(p) = 1, \quad S(p) = 0 \tag{5.11}$$

- Similarly, if z is a RHP-zero of $L(s)$ then

$$T(z) = 0, \quad S(z) = 1 \tag{5.12}$$

Performance Limitations

Sensitivity peaks [5.3.4]:

Maximum modulus principle. *Suppose $f(s)$ is stable (i.e. $f(s)$ is analytic in the complex RHP). Then the maximum value of $|f(s)|$ for s in the right-half plane is attained on the region's boundary, i.e. somewhere along the $j\omega$-axis. Hence, we have for a stable $f(s)$*

$$\|f(j\omega)\|_\infty = \max_\omega |f(j\omega)| \geq |f(s_0)| \quad \forall s_0 \in \mathrm{RHP} \tag{5.13}$$

Performance Limitations

The results below follow from (5.13) with $f(s) = w_P(s)S(s)$, $f(s) = w_T(s)T(s)$ for weighted sensitivity and weighted complementary sensitivity.

Theorem

Weighted sensitivity peak.
Suppose that $G(s)$ has a RHP-zero z and let $w_P(s)$ be any stable weight function. Then for closed-loop stability the weighted sensitivity function must satisfy

$$\|w_P S\|_\infty \geq |w_P(z)| \tag{5.14}$$

- If $w_P(s) = 1$, we have $\|S\|_\infty \geq 1$, which we know must hold since $|S(j\omega)|$ must approach 1 at high frequencies.

Performance Limitations

Theorem

Weighted complementary sensitivity peak.
Suppose that $G(s)$ has a RHP-pole p and let $w_T(s)$ be any stable weight function. Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{5.15}$$

- The basis for this result is that if $G(s)$ has a RHP-pole at $s = p$, then for internal stability $S(p)$ must have a RHP-zero at $s = p$ and from (5.11) we have $T(p) = 1$.
- If $w_T(s) = 1$, we have $\|T\|_\infty \geq 1$, and illustrates that some control is indeed needed to stabilize an unstable plant (since no control, $K = 0$, makes $T = 0$).

Performance Limitations

Theorem

Combined RHP-poles and RHP-zeros.
Suppose that $G(s)$ has N_z RHP-zeros z_j, and N_p RHP-poles p_i. Then for closed-loop stability the weighted sensitivity function must satisfy for each *RHP-zero z_j*

$$\|w_P S\|_\infty \geq c_{1j}|w_P(z_j)|, \quad c_{1j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1 \tag{5.16}$$

and the weighted complementary sensitivity function must satisfy for each *RHP-pole p_i*

$$\|w_T T\|_\infty \geq c_{2i}|w_T(p_i)|, \quad c_{2i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1 \tag{5.17}$$

- If $w_P = w_T = 1$: $\|S\|_\infty \geq \max_j c_{1j}, \quad \|T\|_\infty \geq \max_i c_{2i}$
  $\Longrightarrow$ Large peaks for S and T are unavoidable if a RHP-zero and a RHP-pole are close to each other.

Performance Limitations

Integral Square Error (ISE) optimal control [5.4]

Let us consider the "ideal" controllers that minimizes

$$\text{ISE} = \int_0^\infty |y(t) - r(t)|^2 dt \qquad (5.18)$$

The controller is "ideal" in the sense that if may not be realizable in practice because the cost function includes no penalty on the input $u(t)$. For stable plants with RHP zeros at z_j (real or complex) and a time delay θ, the "ideal" response $y = Tr$ when $r(t)$ is a *unit step* is:

$$T(s) = \prod_i \frac{-s + z_j}{s + \bar{z}_j} e^{-\theta s} \qquad (5.19)$$

where $\bar{z}_j$ is the complex conjugate of z_j.
Optimal ISE for three simple stable plants are:

1. with a delay θ: $\text{ISE} = \theta$
2. with a RHP-zero z: $\text{ISE} = 2/z$
3. with complex RHP-zeros $z = x \pm jy$: $\text{ISE} = 4x/(x^2 + y^2)$

Limitations imposed by time delays [5.5]

Consider a plant $G(s)$ that contains a time delay $e^{-\theta s}$ (and no RHP zeros). Even the "ideal" controller cannot remove this delay (we need to wait). From (5.19), $T = e^{-\theta s}$, and

$$S = 1 - T = 1 - e^{-\theta s} \tag{5.20}$$

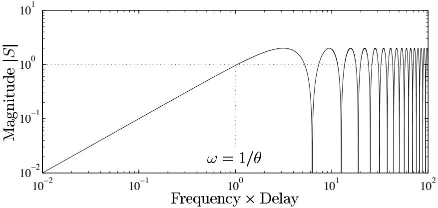


Figure 6: "Ideal" sensitivity function (5.20) for a plant with delay

At low frequency ($\omega\theta < 1$), we have $1 - e^{-\theta s} \approx \theta s$. $|S(j\omega)|$ in Figure 6 crosses 1 at $\frac{\pi}{3}\frac{1}{\theta} = 1.05/\theta$. Because here $|S| = 1/|L|$, we have:

$$\omega_c < 1/\theta \tag{5.21}$$

Performance Limitations

Limitations imposed by RHP-zeros [5.6]

RHP-zeros typically appear when we have competing effects of slow and fast dynamics:

$$G(s) = \frac{1}{s+1} - \frac{2}{s+10} = \frac{-s+8}{(s+1)(s+10)}$$

(a) Inverse response [5.6.1]:

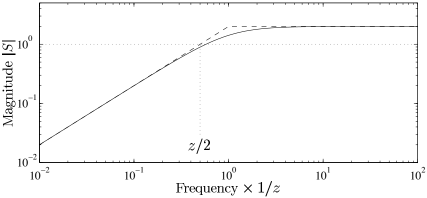
For a stable plant with n_z RHP-zeros, it may be proven that the output in response to a step change in the input will cross zero (its original value) n_z times, that is, we have *inverse response* behaviour.

(a) High-gain instability [5.6.2]:

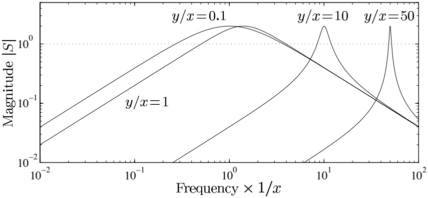
From classical root analysis, as the feedback gain increases towards infinity, the closed-loop poles migrates to the positions of the open-loop RHP zeros $\Rightarrow$ High-gain instability.

Performance Limitations

(b) Bandwidth limitation I [5.6.3]:



(a) Real RHP-zero.



(b) Complex pair of RHP-zeros, $z = x \pm jy$.

Figure 7: "Ideal" sensitivity for plants with RHP-zeros

Performance Limitations

For a single *real RHP-zero* the "ideal", i.e. ISE optimal, sensitivity function is

$$S = 1 - T = \frac{2s}{s+z} \tag{5.22}$$

From Figure 7(a):

$$\omega_B \approx \omega_c < \frac{z}{2} \tag{5.23}$$

Performance Limitations

Bandwidth limitation II [5.6.4]:

Performance requirement:

$$|S(j\omega)| < 1/|w_P(j\omega)| \quad \forall\omega \quad = \quad \|w_P S\|_\infty < 1 \tag{5.24}$$

However, from (5.14) we have that $\|w_P S\|_\infty \geq |w_P(z)|$, so the weight must satisfy

$$\boxed{|w_P(z)| < 1} \tag{5.25}$$

For performance weight

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \tag{5.26}$$

and a real zero at z we get: $\omega_B^*(1 - A) < z\left(1 - \frac{1}{M}\right)$
For example, $A = 0, M = 2$: $\omega_B^* < \frac{z}{2}$

Performance Limitations

Limitations imposed by RHP-poles [5.8]

Specification:

$$|T(j\omega)| < 1/|w_T(j\omega)| \quad \forall\omega \quad = \quad \|w_T T\|_\infty < 1 \tag{5.27}$$

However, from (5.15) we have that:

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{5.28}$$

so the weight must satisfy

$$\boxed{|w_T(p)| < 1} \tag{5.29}$$

Performance Limitations

For:

$$w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T} \tag{5.30}$$

we get:

$$\boxed{\omega_{BT}^* > p\frac{M_T}{M_T - 1}} \tag{5.31}$$

e.g. $M_T = 2$:

$$\omega_{BT}^* > 2p \tag{5.32}$$

Performance Limitations

Combined RHP-poles and RHP-zeros [5.9]

RHP-zero:

$$\omega_c < z/2$$

RHP-pole:

$$\omega_c > 2p$$

RHP-pole and RHP-zero:
$z > 4p$ for acceptable performance and robustness.

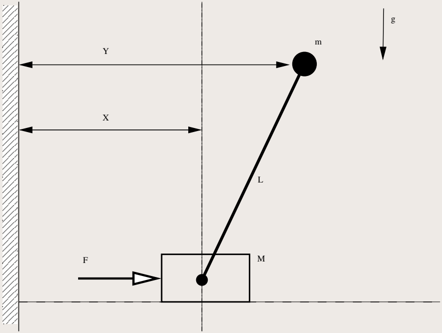
Sensitivity peaks.
From Theorem 6 for a plant with a single real RHP-pole p and a single real RHP-zero z, we always have:

$$\boxed{\|S\|_\infty \geq c,\ \|T\|_\infty \geq c, \quad c = \frac{|z+p|}{|z-p|}} \tag{5.33}$$

Performance Limitations

Example

Balancing a rod. *The objective is to keep the rod upright by movement of the cart, based on observing the rod either at its far end (output y_1) or the cart position (output y_2).*



l *[m] = length of rod*

m *[kg] = mass of rod*

M *[kg] = mass of hand*

$g \approx 10$ *m/s*^{2} *= acceleration due to gravity.*

Performance Limitations

The linearized transfer functions for the two cases are

$$G_1(s) = \frac{-g}{s^2\left(Mls^2 - (M+m)g\right)}; \quad G_2(s) = \frac{ls^2 - g}{s^2\left(Mls^2 - (M+m)g\right)}$$

Poles: $p = 0, 0, \pm\sqrt{\frac{(M+m)g}{Ml}}$. For output $y_1(G_1(s))$ stabilization requires minimum bandwidth (5.32). For output $y_2(G_2(s))$ zero at $z = \sqrt{\frac{g}{l}}$

- For light rod $m << M$, pole $\approx$ zero $\rightarrow$ "impossible" to stabilize
- For heavy rod (m large) difficult to stabilize because $p > z$

Example: $m/M = 0.1 \Rightarrow \|S\|_\infty \geq 42$; $\|T\|_\infty \geq 42 \Rightarrow$ poor control

Non-causal controllers [5.7]

Perfect control can be achieved for a plant with a time delay or RHP-zero if we use a non-causal controller, i.e. a controller which uses information about the future (relevant for servo problems, e.g. in robotics and for batch processing.)

$$G(s) = \frac{-s+z}{s+z}; \quad z > 0 \qquad r(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{5.34}$$

Stable *non-causal controller* generates the input

$$u(t) = \begin{cases} 2e^{zt} & t < 0 \\ 1 & t \geq 0 \end{cases}$$

(See (Figure 8))

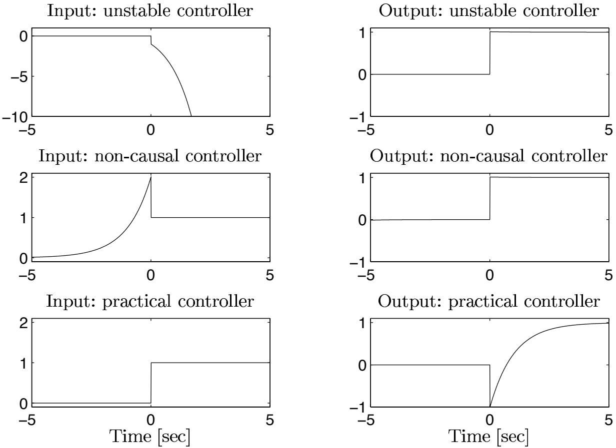


Figure 8: Feedforward control of plant with RHP-zero

Limitations imposed by input constraints [5.11]

For *perfect control* ($e = 0$), the required input is

$$u = G^{-1}r - G^{-1}G_d d \tag{5.35}$$

Disturbance rejection. $r = 0$, $|d(\omega)| = 1$; $|u(\omega)| < 1 \Rightarrow$

$$|G^{-1}(j\omega)G_d(j\omega)| < 1 \iff |G| > |G_d| \quad \forall\omega \tag{5.36}$$

Command tracking. $d = 0$, $|r(\omega)| = R\forall\omega < \omega_r$ $|u(\omega)| < 1 \Rightarrow$

$$|G^{-1}(j\omega)R| < 1 \iff |G| > R \quad \forall\omega \leq \omega_r \tag{5.37}$$

For *acceptable control* (namely $|e(j\omega)| < 1$) we need:

$$\boxed{|G| > |G_d| - 1} \quad \text{at frequencies where} \quad |G_d| > 1 \tag{5.38}$$

$$\boxed{|G| > |R| - 1 < 1} \quad \forall\omega \leq \omega_r \tag{5.39}$$

Summary: Controllability analysis with feedback control [5.14]

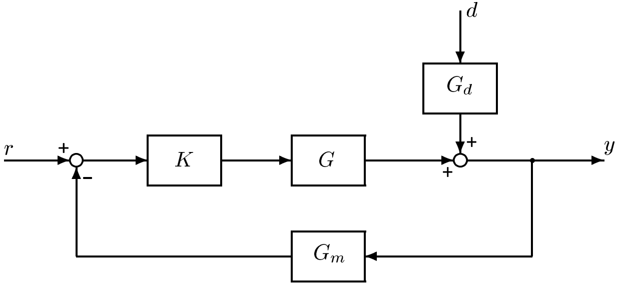


Figure 9: Feedback control system

Performance Limitations

$$y = G(s)u + G_d(s)d; \quad y_m = G_m(s)y \tag{5.40}$$

$G_m(0) = 1$ (perfect steady-state measurement);
d, u, y and r are assumed to be scaled;
$\omega_c =$ gain crossover frequency (frequency where $|L(j\omega)|$ crosses 1 from above);
$\omega_d =$ frequency where $|G_d(j\omega_d)|$ first crosses 1 from above.

Performance Limitations

The following rules apply:

Rule 1. **Speed of response to reject disturbances**. *We require* $\omega_c > \omega_d$. *More specifically,* $|S(j\omega)| \leq |1/G_d(j\omega)| \; \forall\omega$.

Rule 2. **Speed of response to track reference changes**. *We require* $|S(j\omega)| \leq 1/R$ *up to the frequency* ω_r *where tracking is required.*

Rule 3. **Input constraints arising from disturbances**. *For acceptable control (*$|e| < 1$*) we require* $|G(j\omega)| > |G_d(j\omega)| - 1$ *at frequencies where* $|G_d(j\omega)| > 1$.

Rule 4. **Input constraints arising from setpoints**. *We require* $|G(j\omega)| > R - 1$ *up to the frequency* ω_r *where tracking is required.* (See (5.39)).

Performance Limitations

Rule 5. **Time delay θ in $G(s)G_m(s)$**. *We approximately require $\omega_c < 1/\theta$.* (See (5.21)).

Rule 6. **Tight control at low frequencies with a RHP-zero z in $G(s)G_m(s)$**. *For a real RHP-zero we require $\omega_c < z/2$.* (See (5.23)).

Rule 7. **Phase lag constraint**. *We require in most practical cases (e.g. with PID control): $\omega_c < \omega_u$. Here the* ultimate frequency ω_u *is where $\angle GG_m(j\omega_u) = -180°$.*

Rule 8. **Real open-loop unstable pole in $G(s)$ at $s = p$**. *We need high feedback gains to stabilize the system and require $\omega_c > 2p$. In addition, for unstable plants we need $|G| > |G_d|$ up to the frequency p (which may be larger than ω_d where $|G_d| = 1|$).* Otherwise, the input may saturate when there are disturbances, and the plant cannot be stabilized.