Lecture 7

Chapter 9: Controller Design

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7 Controller Design [9]

7.1 Trade-offs in MIMO feedback design [9.1]

Figure 1: One degree-of-freedom feedback

\[
\begin{align*}
y(s) &= T(s)r(s) + S(s)d(s) - T(s)n(s) \\
u(s) &= K(s)S(s) [r(s) - n(s) - d(s)]
\end{align*}
\]
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Closed-loop objectives:

1. For disturbance rejection make $\bar{\sigma}(S)$ small.
2. For noise attenuation make $\bar{\sigma}(T)$ small.
3. For reference tracking make $\bar{\sigma}(T) \approx \sigma(T) \approx 1$.
4. For control energy reduction make $\bar{\sigma}(KS)$ small.
5. For robust stability in the presence of an additive perturbation make $\bar{\sigma}(KS)$ small.
6. For robust stability in the presence of a multiplicative output perturbation make $\bar{\sigma}(T)$ small.

The closed-loop requirements 1 to 6 cannot all be satisfied simultaneously. Feedback design is therefore a trade-off over frequency of conflicting objectives.
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\[(7.3) \quad \sigma(L) - 1 \leq \frac{1}{\sigma(S)} \leq \sigma(L) + 1\]

- At frequencies where \(\sigma(L) \gg 1\), we have \(\sigma(S) \approx 1/\sigma(L)\)
- At frequencies where \(\sigma(L) \ll 1\), we have \(\sigma(T) \approx \sigma(L)\)
- At the bandwidth frequency \((1/\sigma(S(j\omega_B))) = \sqrt{2} = 1/41\), we have \(0.41 \leq \sigma(L(j\omega_B)) \leq 2.41\)
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Over specified frequency ranges, we can approximate the closed-loop requirements by the following open-loop objectives:

1. For *disturbance rejection* make $\sigma(GK)$ large; valid for frequencies at which $\sigma(GK) \gg 1$.

2. For *noise attenuation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

3. For *reference tracking* make $\sigma(GK)$ large; valid for frequencies at which $\sigma(GK) \gg 1$.

4. For *control energy reduction* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

5. For *robust stability to an additive perturbation* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

6. For *robust stability to a multiplicative output perturbation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$. 
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Figure 2: Design tradeoff.
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Requirements 1 and 3 are valid and important at low frequencies, $0 \leq \omega \leq \omega_l \leq \omega_B$. Requirements 2, 4, 5 and 6 are conditions which are valid and important at high frequencies, $\omega_B \leq \omega_h \leq \omega \leq \infty$.

At frequencies where we want high gains (at low frequencies) the “worst-case" direction is related to $\sigma(L)$, whereas at frequencies where we want low gains (at high frequencies) the “worst-case" direction is related to $\bar{\sigma}(L)$.
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7.3 LQG control [9.2]

7.3.1 Traditional LQR and LQG Problems [9.2.1]

Figure 3: Separation Principle
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Figure 4: LQG control configuration
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Figure 5: LQG control configuration with reference
7.3 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control [9.3]

7.3.1 General control problem formulation [9.3.1]

\begin{align*}
\begin{bmatrix}
    z \\
    v
\end{bmatrix} &= P(s) \begin{bmatrix}
    w \\
    u
\end{bmatrix} = \begin{bmatrix}
    P_{11}(s) & P_{12}(s) \\
    P_{21}(s) & P_{22}(s)
\end{bmatrix} \begin{bmatrix}
    w \\
    u
\end{bmatrix} \\
\begin{bmatrix}
    w \\
    u
\end{bmatrix} &= K(s) v
\end{align*}
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The state-space realization of the generalized plant $P$ is given by

$$ P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} $$

(7.8)

$$ z = F_l(P, K)w $$

(7.9)

where

$$ F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} $$

(7.10)

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control involve the minimization of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of $F_l(P, K)$ respectively.
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7.3.2 $\mathcal{H}_2$ optimal control [9.3.2]

The standard $\mathcal{H}_2$ optimal control problem is to find a stabilizing controller $K$ which minimizes

\[
\| F(s) \|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)F(j\omega)^T d\omega}; \quad F \overset{\Delta}{=} F_l(P, K)
\]

For a particular problem the generalized plant $P$ will include the plant model, the interconnection structure, and the designer specified weighting functions. This is illustrated for the LQG problem in the next subsection.

Stochastic interpretation: suppose in the general control configuration that the exogenous input $w$ is white noise of unit intensity. That is:

\[
E \left\{ w(t)w(\tau)^T \right\} = I\delta(t - \tau)
\]
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The expected power in the error signal $z$ is then given by:

\[
\begin{align*}
E \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} z(t)^T z(t) dt \right\} \\
= \text{tr} \ E \left\{ z(t) z(t)^T \right\} \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) F(j\omega)^T d\omega
\end{align*}
\] 
(by Parseval’s Theorem)

\[
\begin{align*}
(7.14) \quad &= \| F \|_2^2 = \| F_l(P, K) \|_2^2
\end{align*}
\]

Thus, by minimizing the $\mathcal{H}_2$ norm, the output (or error) power of the generalized system, due to a unit intensity white noise input, is minimized; we are minimizing the root-mean-square (rms) value of $z$. 

LQG: a special $\mathcal{H}_2$ optimal controller [9.3.3]

\[
\dot{x} = Ax + Bu + w_d \\
y = Cx + w_n
\]

where:

\[
E \left\{ \begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \begin{bmatrix} w_d(\tau)^T \\ w_n(\tau)^T \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t - \tau)
\]

The LQG problem is to find $u = K(s)y$ such that

\[
J = E \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T \begin{bmatrix} x^T Q x + u^T R u \end{bmatrix} dt \right\}
\]

is minimized with $Q = Q^T \geq 0$ and $R = R^T > 0$. 
Define:

\[
(7.19) \quad z = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}
\]

and represent the stochastic inputs \( w_d, w_n \) as

\[
(7.20) \quad \begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{bmatrix} w
\]

where \( w \) is a white noise process of unit intensity. Then the LQG cost function is

\[
(7.21) \quad J = E \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T z(t)^T z(t) dt \right\} = \| F_l(P, K) \|_2^2
\]
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where

\[ z(s) = F_l(P, K)w(s) \]  

(7.22)

and the generalized plant \( P \) is given by

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A & W^{1/2} & 0 & B \\ Q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ C & 0 & V^{1/2} & 0 \end{bmatrix}
\]
Figure 7: The LQG problem: general control configuration
7.3.4 $\mathcal{H}_\infty$ optimal control [9.3.4]

With reference to the general control configuration of Figure 6, the standard $\mathcal{H}_\infty$ optimal control problem is to find all stabilizing controllers $K$ which minimize

$$
\|F_l(P, K)\|_\infty = \max_\omega \sigma(F_l(P, K)(j\omega))
$$

(7.23)

This has a time domain interpretation as the induced (worst-case) $2$-norm. Let $z = F_l(P, K)w$, then

$$
\|F_l(P, K)\|_\infty = \max_{w(t) \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}
$$

(7.24)

where $\|z(t)\|_2 = \sqrt{\int_0^\infty \sum_i |z_i(t)|^2 dt}$ is the $2$-norm of the vector signal.
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It is often computationally (and theoretically) simpler to design a sub-optimal one (i.e. one close to the optimal controller in the sense of the $\mathcal{H}_\infty$ norm). Let $\gamma_{\text{min}}$ be the minimum value of $\|F_l(P, K)\|_\infty$ over all stabilizing controllers $K$. Then the $\mathcal{H}_\infty$ sub-optimal control problem is: given a $\gamma > \gamma_{\text{min}}$, find all stabilizing controllers $K$ such that

$$\|F_l(P, K)\|_\infty < \gamma$$
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7.3.5 Mixed-sensitivity $\mathcal{H}_\infty$ control [9.3.5]

To optimize performance, minimize $\|w_1S\|_\infty$, to minimize control inputs, minimize $\|w_2KS\|_\infty$.
Compromise:

$$\| \begin{bmatrix} w_1S \\ w_2KS \end{bmatrix} \|_\infty$$ (7.25)

General setting: disturbance $d$ as a single exogenous input, error signal $z = \begin{bmatrix} z_1^T \\ z_2^T \end{bmatrix}^T$, where $z_1 = W_1y$ and $z_2 = -W_2u$, (8).
Figure 8: $S/KS$ mixed-sensitivity optimization in standard form (regulation)
Thus \( z_1 = W_1 Sw \) and \( z_2 = W_2 KS w \) and:

\[
\begin{align*}
P_{11} &= \begin{bmatrix} W_1 \\ 0 \end{bmatrix} & P_{12} &= \begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix} \\
P_{21} &= -I & P_{22} &= -G
\end{align*}
\]

(7.26)

where the partitioning is such that

\[
\begin{bmatrix} z_1 \\ z_2 \\ - - - \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}
\]

(7.27)

and

\[
F_l(P, K) = \begin{bmatrix} W_1 S \\ W_2 KS \end{bmatrix}
\]

(7.28)
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Another useful mixed sensitivity optimization problem, is to find a stabilizing controller which minimizes

$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty$$

(7.29)

The $S/T$ mixed-sensitivity minimization problem can be put into the standard control configuration as shown in Figure 9.

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} -W_1 G \\ W_2 G \end{bmatrix}$$

$$P_{21} = I, \quad P_{22} = -G$$

(7.30)
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Figure 9: $S/T$ mixed-sensitivity optimization in standard form
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7.4 $\mathcal{H}_\infty$ loop-shaping design [9.4]

We need a design procedure more flexible than mixed-sensitivity $\mathcal{H}_\infty$ but not as complicated as $\mu$-synthesis. For simplicity, it should be based on classical loop-shaping ideas.

7.4.1 Coprime Factorization [4.1.5]

A useful way to represent systems is the coprime factorization, which may be used both in state-space and transfer function forms.

A right coprime factorization of $G$ is given by

\begin{equation}
G(s) = N_r(s)M_r^{-1}(s)
\end{equation}

(7.31)

where $N_r(s)$ and $M_r$ are stable coprime transfer functions.
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The stability implies that:

- $N_r(s)$ should contain all the RHP-zeros of $G(s)$
- $M_r$ should contain as RHP-zeros all the RHP-poles of $G(s)$

The coprimeness implies that:

- there should be no common RHP-zeros (including the point at infinity) in $N_r(s)$ and $M_r$, which results in pole-zero cancellation when forming $N_r(s)M_r^{-1}(s)$.

Mathematically, comprimeness means that there exist stable $U_r(s)$ and $V_r(s)$ such that the following Bezout identity is satisfied:

(7.32) \[ U_r N_r + V_r M_r = I \]
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Similarly, a *left coprime factorization* of $G$ is given by

(7.33) \[ G(s) = M_l^{-1}(s)N_l(s) \]

where $N_l(s)$ and $M_l$ are stable coprime transfer functions. That is, there exist stable $U_l(s)$ and $V_l(s)$ such that the following Bezout identity is satisfied:

(7.34) \[ N_lU_l + M_lV_l = I \]

**Example**

(7.35) \[ G(s) = \frac{(s - 1)(s + 2)}{(s - 3)(s + 4)} \]
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The coprime factorization is NOT unique. We introduce the operator $M^*$ defined as $M^*(s) = M^T(-s)$

Then, $G(s) = N_r(s)M_r^{-1}(s)$ is called a normalized right coprime factorization if

$$N_r^*N_r + M_r^*M_r = I \quad (7.36)$$

In this case, $X_r(s) = \begin{bmatrix} M_r \\ N_r \end{bmatrix}$ is an inner transfer function which means that $X_r^*X_r = I$.

Then, $G(s) = M_l^{-1}(s)N_l(s)$ is called a normalized left coprime factorization if

$$N_lN_l^* + M_lM_l^* = I \quad (7.37)$$

In this case, $X_l(s) = [M_l \quad N_l]$ is a co-inner transfer function which means that $X_lX_l^* = I$. 
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If $G$ has a minimal state-space realization

$$G \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ (7.38)

Then, a minimal state-space representation of a normalized left coprime factorization is given by

$$\begin{bmatrix} N_l(s) & M_l(s) \end{bmatrix} \equiv \begin{bmatrix} A + HC & B + HD & H \\ R^{-\frac{1}{2}}C & R^{-\frac{1}{2}}D & R^{-\frac{1}{2}} \end{bmatrix}$$ (7.39)

where $H = -(BD^T + ZC^T)R^{-1}$, $R = I + DD^T$, and the matrix $Z$ is the unique positive definite solution to the Riccati equation

$$(A - BS^{-1}D^T C)Z + Z(A - BS^{-1}D^T C)^T - ZC^T R^{-1} CZ + BS^{-1} B^T = 0$$

where $S = I + D^T D$. 

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7.4.2 RS for Coprime Factor Uncertainty [8.6.2]

\[(7.40) \quad \text{RS} \iff \| M \|_\infty < 1\]

is tight (not conservative) only when there is a single full perturbation block. An “exception" to this is when the uncertainty blocks enter or exit from the same location in the block diagram, because they can be stacked on top of each other or side by side in an overall $\Delta$ which is then a full matrix. If we norm-bound the combined (stacked) uncertainty, we then get a tight RS condition in terms of $\| M \|_\infty$. 
One important uncertainty that falls into this category is the coprime uncertainty, for which the set of plants is

\begin{equation}
G_p = (M_l + \Delta_M)^{-1}(N_l + \Delta_N), \quad \|\Delta_N \ \Delta_M\|_\infty \leq \epsilon
\end{equation}

where $G = M_l^{-1}N_l$ is a left coprime factorization of the nominal plant.
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To test for RS we can rearrange the block diagram into the $M\Delta$ structure with

$$\Delta = [\Delta_N \quad \Delta_M]; \quad M = \begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} M_l^{-1}$$

We then have

$$\text{RS} \quad \forall \|\Delta_N \quad \Delta_M\|_\infty \leq \epsilon \iff \|M\|_\infty < 1/\epsilon$$

(7.43)

This result is central to the $\mathcal{H}_\infty$ loop-shaping design procedure.

- Good “generic” uncertainty description when no a-priori uncertainty information is available.
- Often used to maximize the uncertainty magnitude $\epsilon$ such that RS is maintained.
7.4.3 Robust Stabilization [9.4.1]

For feedback systems with coprime uncertainty, the stability property is robust if and only if the nominal feedback is stable and

\[
\gamma_K = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} M_l^{-1} \right\|_\infty < 1/\epsilon \quad \forall \| \Delta_N \Delta_M \|_\infty \leq \epsilon
\]

(7.44)

The lowest achievable \( \gamma_K \) and the corresponding maximum stability margin \( \epsilon \) were computed analytically

\[
\gamma_{min} = \epsilon_{max} = \left\{ 1 - \| [N_l \ M_l] \|_H^2 \right\}^{-1/2} = (1 + \rho(XZ))^{1/2}
\]

(7.45)

where \( \| \cdot \|_H \) denotes Hankel norm and \( \rho \) denotes the spectral radius (maximum eigenvalue).
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For a minimal realization \((A, B, C, D)\) of \(G(s)\), \(Z\) is the unique positive definite solution to the algebraic Riccati equation

\[
(A - BS^{-1}D^TC)Z + Z(A - BS^{-1}D^TC)^T - ZC^TR^{-1}CZ + BS^{-1}B^T = 0
\]

(7.46)

where

\[
R = I + DD^T, \quad S = I + D^TD
\]

(7.47)

and \(X\) is the unique positive definite solution to the algebraic Riccati equation

\[
(A - BS^{-1}D^TC)^TX + X(A - BS^{-1}D^TC) - XBS^{-1}B^TX + C^TR^{-1}C = 0
\]

(7.48)

This formula simplifies considerably for a strictly proper plant, i.e., when \(D=0\);
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A controller that guarantees that

\[ \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} M_l^{-1} \right\|_{\infty} \leq \gamma \]

for a specified \(\gamma > \gamma_{\text{min}}\), is given by

\[
K \overset{s}{=} \begin{bmatrix}
A + BF + \gamma^2 (L^T)^{-1} ZC^T (C + DF) & \gamma^2 (L^T)^{-1} ZC^T \\
B^T X & -D^T
\end{bmatrix}
\]

\[
F = -S^{-1} (D^T C + B^T X)
\]

\[
L = (1 - \gamma^2) I + X Z
\]

Since we can compute directly \(\gamma_{\text{min}}\), we get an explicit solution by solving just two Riccati equations and avoid the \(\gamma\)-iteration needed to solve the general \(\mathcal{H}_\infty\) problem.
7.4.4 A Systematic $\mathcal{H}_\infty$ Loop-Shaping Procedure [9.4.2]

Robust stabilization alone is not much use in practice because the designer is not able to specify any performance requirements. We can add pre- and post-compensators to the plant to shape the open-loop singular values prior to robust stabilization of the “shaped” plant.

Figure 11: Shaped plant and controller
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If $W_1$ and $W_2$ are the pre- and post-compensators respectively, the “shaped” plant (initial loop shape) $G_s$ is given by

$$G_s = W_2 GW_1$$

(7.50)

The controller $K_s$ is synthesized by solving the robust stabilization problem for the “shaped” plant with a normalized left coprime factorization $G_s = M_s^1 N_s$. The feedback controller for the plant $G$ is then

$$K = W_1 K_s W_2$$

(7.51)

This procedure contains all the essential ingredients of classical loop-shaping. The robust stabilization problem can be solved using the formulae presented in the previous section.