#### Multivariable Robust Control

Lecture 7 (Meetings 19-20) Chapter 9: Controller Design

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#### Trade-offs in MIMO feedback design [9.1]:



Figure 1: One degree-of-freedom feedback

The closed-loop transfer function are written as

$$y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s)$$
(6.1)

$$u(s) = K(s)S(s)[r(s) - n(s) - d(s)]$$
(6.2)

where  $S = (I + GK)^{-1}$  and  $T = GK(I + GK)^{-1}$ .

Closed-loop objectives:

- For disturbance rejection make  $\bar{\sigma}(S)$  small.
- **②** For noise attenuation make  $\bar{\sigma}(T)$  small.
- For reference tracking make  $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$ .
- For control energy reduction make  $\bar{\sigma}(KS)$  small.
- § For robust stability in presence of additive perturbation  $\rightarrow \bar{\sigma}(KS)$  small.
- **③** For robust stability in presence of multiplicative output perturbation  $\rightarrow \bar{\sigma}(T)$  small.

The closed-loop requirements 1 to 6 cannot all be satisfied simultaneously.

Feedback design is therefore a trade-off over frequency of conflicting objectives.

But this is not always as difficult as it looks like because the frequency ranges over which the objectives are important can be quite different. For example, disturbance rejection is typically a low-frequency requirement, while noise attenuation is often only relevant at high frequencies.

In classical loop shaping, it is the loop gain L = GK what is shaped. However,

$$\underline{\sigma}(L) - 1 \le \frac{1}{\overline{\sigma}(S)} \le \underline{\sigma}(L) + 1$$
(6.3)

- At frequencies where  $\underline{\sigma}(L)>>1$ , we have  $\bar{\sigma}(S)\approx 1/\underline{\sigma}(L)$
- At frequencies where  $\bar{\sigma}(L) << 1$ , we have  $\bar{\sigma}(T) \approx \bar{\sigma}(L)$
- At the bandwidth frequency  $(1/\bar{\sigma}(S(j\omega_B)) = \sqrt{2} = 1/41)$ , we have  $0.41 \le \underline{\sigma}(L(j\omega_B)) \le 2.41$

Over specified frequency ranges, we can approximate the closed-loop requirements by the following open-loop objectives:

- For disturbance rejection make  $\underline{\sigma}(GK)$  large; valid for frequencies at which  $\underline{\sigma}(GK) \gg 1$ .
- **②** For noise attenuation make  $\bar{\sigma}(GK)$  small; valid for frequencies at which  $\bar{\sigma}(GK) \ll 1$ .
- For reference tracking make  $\underline{\sigma}(GK)$  large; valid for frequencies at which  $\underline{\sigma}(GK) \gg 1$ .
- For control energy reduction make  $\bar{\sigma}(K)$  small; valid for frequencies at which  $\bar{\sigma}(GK) \ll 1$ .
- For robust stability to an additive perturbation make  $\bar{\sigma}(K)$  small; valid for frequencies at which  $\bar{\sigma}(GK) \ll 1$ .
- For robust stability to a multiplicative output perturbation make  $\bar{\sigma}(GK)$  small; valid for frequencies at which  $\bar{\sigma}(GK) \ll 1$ .



Figure 2: Design tradeoff.

- Requirements 1 and 3 are valid and important at low frequencies,  $0 \le \omega \le \omega_l \le \omega_B$ .
- Requirements 2, 4, 5 and 6 are conditions which are valid and important at high frequencies,  $\omega_B \leq \omega_h \leq \omega \leq \infty$ .
- At frequencies where we want high gains (at low frequencies) the "worst-case" direction is related to  $\underline{\sigma}(L)$ , whereas at frequencies where we want low gains (at high frequencies) the "worst-case" direction is related to  $\overline{\sigma}(L)$ .

#### LQG control [9.2]

Traditional LQR and LQG Problems [9.2.1]:

$$\dot{x} = Ax + Bu + w_d \tag{6.4}$$

$$y = Cx + w_n \tag{6.5}$$

where (process/measurement noises are uncorrelated zero-mean Gaussian stochastic processes with constant power spectral density matrices W and V)

$$E\left\{\begin{bmatrix} w_d(t)\\ w_n(t)\end{bmatrix}\begin{bmatrix} w_d(\tau)^T & w_n(\tau)^T\end{bmatrix}\right\} = \begin{bmatrix} W & 0\\ 0 & V\end{bmatrix}\delta(t-\tau)$$
(6.6)

The LQG problem is to find u = K(s)y such that

$$J = E\left\{\lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ x^T Q x + u^T R u \right] dt \right\}$$
(6.7)

is minimized with  $Q=Q^T\geq 0$  and  $R=R^T>0.$ 

The solution of the LQG problem, thanks to the Separation Principle, is surprisingly simple and elegant. As illustrated in the figure below, the LQG problem and its solution can be separated into two distinct parts.



Figure 3: Separation Principle

**Optimal State Feedback.** The LQR problem, where all the states are known, is the deterministic initial value problem: given the system  $\dot{x} = Ax + Bu$  with a non-zero initial state x(0), find the input signal u(t) which takes the system to the zero state (x = 0) in an optimal manner, i.e. by minimizing the deterministic cost

$$J_r = \int_0^\infty \left( x(t)^T Q x(t) + u(t)^T R u(t) \right) dt$$
 (6.8)

The optimal solution is  $u(t) = -K_r x(t)$ , where  $K_r = R^{-1}B^T X$  and  $X = X^T \ge 0$  is the unique positive semi-definite solution of the algebraic Riccati equation

$$A^{T}X + XA - XBR^{-1}B^{T}X + Q = 0.$$
 (6.9)

**Kalman Filter.** The Kalman filter has the structure of an ordinary state estimator or observer, as shown in Figure 7, with

$$\dot{\widehat{x}} = A\widehat{x} + Bu + K_f(y - C\widehat{x}) \tag{6.10}$$

The optimal solution choice of  $K_f$ , which minimizes  $E\{[x - \hat{x}]^T [x - \hat{x}]\}$ , is given by  $K_f = YC^TV^{-1}$  where  $Y = Y^T \ge 0$  is the unique positive semi-definite solution of the algebraic Riccati equation

$$Y^{T}A + AY - YC^{T}V^{-1}CY + W = 0.$$
(6.11)



#### Figure 4: LQG control configuration

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Figure 5: LQG control configuration with reference

The controllers is arranged to show its two-degrees-of-freedom structure. What is the transfer function linking  $[r^T \ y^T]^T$  to u? What is the steady-state solution if the closed-loop is stable?

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#### Example: LQG design with integral action for inverse response process.

The standard LQG design procedure does not give a controller with integral action (see Figure 5). Therefore, the setup must be augmented with an integrator before designing the state feedback regulator. The plant is given by

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$$
(6.12)

We choose Q such that only the integrated state y - r is weighted and we choose R = 1 (only the ratio between Q and R matters and reducing R yields a faster response). The Kalman fiter is set up so that the integrated states are not estimated. We select W = wI (process noise directly on the states) with w = 1 and we choose V = 1 (measurement noise) (only the ratio between w and V matters and reducing V we put more trust in the measurement).



Figure 6: LQG control configuration with integral action and reference input



Figure 7: Closed-loop response to unit step in r.

The time response is good and similar to that obtained by a loop-shaping design in Lecture 2.

#### $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control [9.3]

General control problem formulation [9.3.1]:



Figure 8: General control configuration

$$\begin{bmatrix} z \\ v \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$
(6.13)  
$$u = K(s)v$$
(6.14)

The state-space realization of the generalized plant P is given by

$$P \stackrel{s}{=} \begin{bmatrix} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$
(6.15)  
$$z = F_l(P, K)w$$
(6.16)

where

$$F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$
(6.17)

 $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control involve the minimization of the  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norms of  $F_l(P,K)$  respectively.

#### $\mathcal{H}_2$ optimal control [9.3.2]:

The standard  $\mathcal{H}_2$  optimal control problem is to find a stabilizing controller K which minimizes

$$\| F(s)\|_{2} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} \left(F(j\omega)F(j\omega)^{T}\right) d\omega}; \quad F \stackrel{\triangle}{=} F_{l}(P, K)$$
(6.18)

- For a particular problem the generalized plant P will include the plant model, the interconnection structure, and the designer specified weighting functions. This is illustrated for the LQG problem in the next subsection.
- Stochastic interpretation: suppose in the general control configuration that the exogenous input *w* is white noise of unit intensity. That is:

$$E\left\{w(t)w(\tau)^{T}\right\} = I\delta(t-\tau)$$
(6.19)

• The expected power in the error signal z is then given by:

$$E\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}z(t)^{T}z(t)dt\right\}$$

$$= \operatorname{tr} E\left\{z(t)z(t)^{T}\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{tr}\left(F(j\omega)F(j\omega)^{T}\right)d\omega$$
(by Parseval's Theorem)
$$= \|F\|_{2}^{2} = \|F_{l}(P,K)\|_{2}^{2}$$
(6.21)

• Thus, by minimizing the  $\mathcal{H}_2$  norm, the output (or error) power of the generalized system, due to a unit intensity white noise input, is minimized; we are minimizing the root-mean-square (rms) value of z.

# Uncertainty in MIMO Systems

#### Theorem

**Parseval's Theorem:** The sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform, i.e.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( X(\omega) X(\omega)^T \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$
(6.22)

where  $X(\omega) = \mathcal{F}_{\omega}\{x(t)\}$  represents the continuous Fourier transform of x(t). The interpretation of this form of the theorem is that the total energy of a signal can be calculated by summing power-per-sample across time or, equivalently, by summing the spectral power across frequency.

LQG: a special  $\mathcal{H}_2$  optimal controller [9.3.3]:

$$\dot{x} = Ax + Bu + w_d \tag{6.23}$$

$$y = Cx + w_n \tag{6.24}$$

where:

$$E\left\{\begin{bmatrix}w_d(t)\\w_n(t)\end{bmatrix}\begin{bmatrix}w_d(\tau)^T & w_n(\tau)^T\end{bmatrix}\right\} = \begin{bmatrix}W & 0\\0 & V\end{bmatrix}\delta(t-\tau)$$
(6.25)

The LQG problem is to find u = K(s)y such that

$$J = E\left\{\lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ x^T Q x + u^T R u \right] dt \right\}$$
(6.26)

is minimized with  $Q = Q^T \ge 0$  and  $R = R^T > 0$ .

Define:

$$z = \begin{bmatrix} Q^{\frac{1}{2}} & 0\\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x\\ u \end{bmatrix}$$
(6.27)

and represent the stochastic inputs  $w_d$ ,  $w_n$  as

$$\begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} & 0 \\ 0 & V^{\frac{1}{2}} \end{bmatrix} w$$
(6.28)

where w is a white noise process of unit intensity. Then the LQG cost function is

$$J = E\left\{\lim_{T \to \infty} \frac{1}{T} \int_0^T z(t)^T z(t) dt\right\} = \|F_l(P, K)\|_2^2$$
(6.29)

where

$$z(s) = F_l(P, K)w(s)$$
(6.30)

and the generalized plant  $\boldsymbol{P}$  is given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{\mathsf{s}}{=} \begin{bmatrix} A & W^{\frac{1}{2}} & 0 & B \\ Q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{\frac{1}{2}} \\ C & 0 & V^{\frac{1}{2}} & 0 \end{bmatrix}$$
(6.31)



Figure 9: The LQG problem: general control configuration

#### $\mathcal{H}_{\infty}$ optimal control [9.3.4]:

With reference to the general control configuration of Figure 8, the standard  $\mathcal{H}_{\infty}$  optimal control problem is to find all stabilizing controllers K which minimize

$$\|F_l(P,K)\|_{\infty} = \max_{\omega} \bar{\sigma}(F_l(P,K)(j\omega))$$
(6.32)

This has a time domain interpretation as the induced (worst-case) 2-norm. Let  $z = F_l(P, K)w$ , then

$$\|F_l(P,K)\|_{\infty} = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$
(6.33)

where  $||z(t)||_2 = \sqrt{\int_0^\infty \sum_i |z_i(t)|^2 dt}$  is the 2-norm of the vector signal.

- It is often computationally (and theoretically) simpler to design a sub-optimal one (i.e. one close to the optimal controller in the sense of the  $\mathcal{H}_{\infty}$  norm).
- Let  $\gamma_{\min}$  be the minimum value of  $\|F_l(P,K)\|_{\infty}$  over all stabilizing controllers K.
- Then the  $\mathcal{H}_{\infty}$  sub-optimal control problem is: given a  $\gamma > \gamma_{\min}$ , find all stabilizing controllers K such that

 $\|F_l(P,K)\|_{\infty} < \gamma$ 

Mixed-sensitivity  $\mathcal{H}_{\infty}$  control [9.3.5]:

- $\bullet\,$  To optimize performance, minimize  $\|w_1S\|_\infty$
- To minimize control inputs, minimize  $\|w_2KS\|_\infty$
- Compromise: Minimize

$$\left\| \begin{bmatrix} w_1 S \\ w_2 KS \end{bmatrix} \right\|_{\infty} \tag{6.34}$$

• General setting (Fig. 10): disturbance d as a single exogenous input, error signal  $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$ , where  $z_1 = W_1 y$  and  $z_2 = -W_2 u$ .



Figure 10: S/KS mixed-sensitivity optimization in standard form (regulation)

Thus  $z_1 = W_1 S w$  and  $z_2 = W_2 K S w$  and:

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix}$$

$$P_{21} = -I \qquad P_{22} = -G$$
(6.35)

where the partitioning is such that

$$\begin{bmatrix} z_1\\ z_2\\ -\frac{1}{v} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12}\\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w\\ u \end{bmatrix}$$
(6.36)

and

$$F_l(P,K) = \begin{bmatrix} W_1 S \\ W_2 KS \end{bmatrix}$$
(6.37)

Another useful mixed sensitivity optimization problem, is to find a stabilizing controller which minimizes

$$\left\| \left[ \begin{array}{c} W_1 S \\ W_2 T \end{array} \right] \right\|_{\infty} \tag{6.38}$$

The S/T mixed-sensitivity minimization problem can be put into the standard control configuration as shown in Figure 11.

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} -W_1G \\ W_2G \end{bmatrix}$$

$$P_{21} = I \qquad P_{22} = -G$$
(6.39)



Figure 11: S/T mixed-sensitivity optimization in standard form

#### $\mathcal{H}_{\infty}$ loop-shaping design [9.4]:

We need a design procedure more flexible than mixed-sensitivity  $\mathcal{H}_\infty$  but not as complicated as  $\mu$ -synthesis. For simplicity, it should be based on classical loop-shaping ideas.

Coprime Factorization [4.1.5]:

A useful way to represent systems is the coprime factorization, which may be used both in state-space and tranfer function forms.

A right coprime factorization of G is given by

$$G(s) = N_r(s)M_r^{-1}(s)$$
(6.40)

where  $N_r(s)$  and  $M_r$  are stable coprime transfer functions.

The stability implies that:

- $\bullet \ N_r(s)$  should contain all the RHP-zeros of G(s)
- $M_r$  should contain as RHP-zeros all the RHP-poles of G(s)

The coprimeness implies that:

• there should be no common RHP-zeros (including the point at infinity) in  $N_r(s)$  and  $M_r$ , which results in pole-zero cancellation when forming  $N_r(s)M_r^{-1}(s)$ .

Mathematically, comprimeness means that there exist stable  $U_r(s)$  and  $V_r(s)$  such that the following Bezout identity is satisfied:

$$U_r N_r + V_r M_r = I \tag{6.41}$$

Similarly, a *left coprime factorization* of G is given by

$$G(s) = M_l^{-1}(s)N_l(s)$$
(6.42)

where  $N_l(s)$  and  $M_l$  are stable coprime transfer functions. That is, there exist stable  $U_l(s)$  and  $V_l(s)$  such that the following Bezout identity is satisfied:

$$N_l U_l + M_l V_l = I \tag{6.43}$$

Example

$$G(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

$$N(s) = \frac{(s-1)}{(s+4)}, M(s) = \frac{(s-3)}{(s+2)}$$
(6.44)

- $\bullet\,$  The coprime factorization is NOT unique. We introduce the operator  $E^*$  defined as  $E^*(s)=E^T(-s)$
- $\bullet\,$  Then,  $G(s)=N_r(s)M_r^{-1}(s)$  is called a normalized right coprime factorization if

$$N_r^* N_r + M_r^* M_r = I (6.45)$$

- In this case,  $X_r(s) = \begin{bmatrix} M_r \\ N_r \end{bmatrix}$  is a *inner* transfer function which means that  $X_r^*X_r = I.$
- $\bullet\,$  Then,  $G(s)=M_l^{-1}(s)N_l(s)$  is called a normalized left coprime factorization if

$$N_l N_l^* + M_l M_l^* = I (6.46)$$

• In this case,  $X_l(s) = [M_l \ N_l]$  is a *co-inner* transfer function which means that  $X_l X_l^* = I$ .

#### Example

$$G(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

$$N(s) = k \frac{(s-1)(s+2)}{(s^2+k_1s+k_2)}, M(s) = k \frac{(s-3)(s+4)}{(s^2+k_1s+k_2)}$$
(6.47)

for any k and any  $k_1, k_2 > 0$ .

Substituting in (6.45), after some algebra we can obtain  $k = \pm 0.71$ ,  $k_1 = 5.67$ , and  $k_2 = 8.6$ .

 $\bullet~$  If G has a minimal state-space realization

$$G \stackrel{\mathsf{s}}{=} \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{6.48}$$

• Then, a minimal state-space representation of a normalized left coprime factorization is given by

$$[N_l(s)M_l(s)] \stackrel{s}{=} \left[ \begin{array}{c|c} A + HC & B + HD & H \\ \hline R^{-\frac{1}{2}}C & R^{-\frac{1}{2}}D & R^{-\frac{1}{2}} \end{array} \right]$$
(6.49)

where  $H = -(BD^T + ZC^T)R^{-1}$ ,  $R = I + DD^T$ , and the matrix Z is the unique positive definite solution to the Riccati equation

$$(A - BS^{-1}D^{T}C)Z + Z(A - BS^{-1}D^{T}C)^{T} - ZC^{T}R^{-1}CZ + BS^{-1}B^{T} = 0$$

where  $S = I + D^T D$ .

RS for Coprime Factor Uncertainty [8.6.2]:

• The condition

$$\mathsf{RS} \iff \|M\|_{\infty} < 1 \tag{6.50}$$

is tight (not conservative) only when there is a single full perturbation block.

- An "exception" to this is when the uncertainty blocks enter or exit from the same location in the block diagram, because they can be stacked on top of each other or side by side in an overall  $\Delta$  which is then a full matrix.
- If we norm-bound the combined (stacked) uncertainty, we then get a tight RS condition in terms of  $\|M\|_{\infty}$ .

One important uncertainty that falls into this category is the coprime uncertainty, for which the set of plants is

$$G_p = (M_l + \Delta_M)^{-1} (N_l + \Delta_N), \quad \|\Delta_N \quad \Delta_M\|_{\infty} \le \epsilon$$
(6.51)

where  $G = M_l^{-1} N_l$  is a left coprime factorization of the nominal plant.



Figure 12: Coprime uncertainty

To test for RS we can rearrange the block diagram into the  $M\Delta$  structure with

$$\Delta = \begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix}; \qquad M = \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} M_l^{-1}$$
(6.52)

We then have

$$\mathsf{RS} \ \forall \|\Delta_N \quad \Delta_M\|_{\infty} \le \epsilon \iff \|M\|_{\infty} < 1/\epsilon \tag{6.53}$$

This result is central to the  $\mathcal{H}_\infty$  loop-shaping design procedure.

- Good "generic" uncertainty description when no a-priori uncertainty information is available.
- $\bullet$  Often used to maximize the uncertainty magnitude  $\epsilon$  such that RS is maintained.

#### Robust Stabilization [9.4.1]:

For feedback systems with coprime uncertainty, the stability property is robust *if* and only *if* the nominal feedback is stable and

$$\gamma_{K} = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} M_{l}^{-1} \right\|_{\infty} < 1/\epsilon \quad \forall \|\Delta_{N} \quad \Delta_{M}\|_{\infty} \le \epsilon$$
(6.54)

The lowest achievable  $\gamma_K$  and the corresponding maximum stability margin  $\epsilon$  were computed analytically

$$\gamma_{min} = \epsilon_{max}^{-1} = \left\{ 1 - \| [N_l \ M_l] \|_H^2 \right\}^{-1/2} = (1 + \rho(XZ))^{1/2}$$
(6.55)

where  $\|\cdot\|_H$  denotes Hankel norm and  $\rho$  denotes the spectral radius (maximum eigenvalue).

For a minimal realization (A, B, C, D) of G(s), Z is the unique positive definite solution to the algebraic Riccati equation

$$(A - BS^{-1}D^{T}C)Z + Z(A - BS^{-1}D^{T}C)^{T} - ZC^{T}R^{-1}CZ + BS^{-1}B^{T} = 0$$
(6.56)

where

$$R = I + DD^T, \qquad S = I + D^T D \tag{6.57}$$

and X is the unique positive definite solution to the algebraic Riccati equation

$$(A - BS^{-1}D^{T}C)^{T}X + X(A - BS^{-1}D^{T}C) - XBS^{-1}B^{T}X + C^{T}R^{-1}C = 0$$
(6.58)

This formula simplifies considerably for a strictly proper plant, i.e., when D=0;

A controller that guarantees that

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} M_l^{-1} \right\|_{\infty} \le \gamma$$
(6.59)

for a specified  $\gamma > \gamma_{min}$ , is given by

$$K \stackrel{\mathbf{s}}{=} \left[ \begin{array}{c|c} A + BF + \gamma^2 (L^T)^{-1} Z C^T (C + DF) & \gamma^2 (L^T)^{-1} Z C^T \\ \hline B^T X & & -D^T \end{array} \right]$$
$$F = -S^{-1} (D^T C + B^T X)$$
$$L = (1 - \gamma^2) I + X Z$$

Since we can compute directly  $\gamma_{min}$ , we get an explicit solution by solving just two Riccati equations and avoid the  $\gamma$ -iteration needed to solve the general  $\mathcal{H}_{\infty}$  problem.

#### A Systematic $\mathcal{H}_{\infty}$ Loop-Shaping Procedure [9.4.2]:

Robust stabilization alone is not of much use in practice because the designer is not able to specify any performance requirements. We can add pre- and post-compensators to the plant to shape the open-loop singular values prior to robust stabilization of the "shaped" plant.



Figure 13: Shaped plant and controller

If  $W_1$  and  $W_2$  are the pre- and post-compensators respectively, the "shaped" plant (initial loop shape)  $G_s$  is given by

$$G_s = W_2 G W_1 \tag{6.60}$$

The controller  $K_s$  is synthesized by solving the robust stabilization problem for the "shaped" plant with a normalized left coprime factorization  $G_s = M_s^{-1}N_s$ . The feedback controller for the plant G is then

$$K = W_1 K_s W_2 \tag{6.61}$$

This procedure contains all the essential ingredients of classical loop-shaping. The robust stabilization problem can be solved using the formulae presented in the previous section.