Multivariable Robust Control

Lecture 6 (Meetings 13-14)

Chapter 8: Robust Stability and Performance Analysis for MIMO Systems

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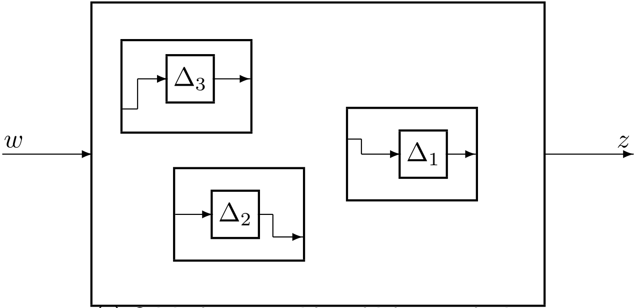
General configuration with uncertainty [8.1]

For our robustness analysis we use a system representation in which the uncertain perturbations are "pulled out" into a block-diagonal matrix,

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix} \tag{6.1}$$

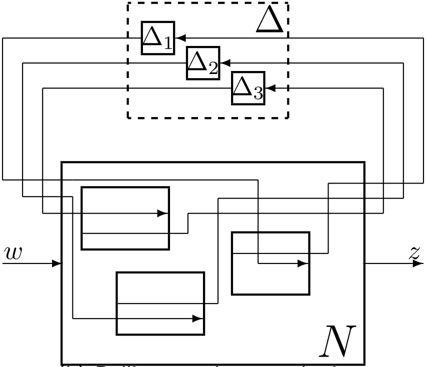
where each Δ_i represents a specific source of uncertainty.

Uncertainty in MIMO Systems



(a) Original system with multiple perturbations

Uncertainty in MIMO Systems



(b) Pulling out the perturbations

Figure 1: Rearranging an uncertain system into the $N\Delta$-structure

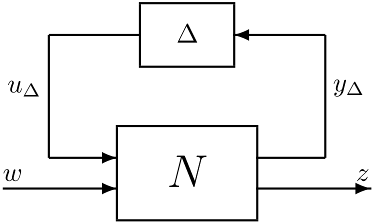


Figure 2: $N\Delta$-structure for robust performance analysis

If we also pull out the controller K, we get the generalized plant P, as shown in Figure 3. *For analysis* of the uncertain system, we use the $N\Delta$-structure in Figure 2.

Uncertainty in MIMO Systems

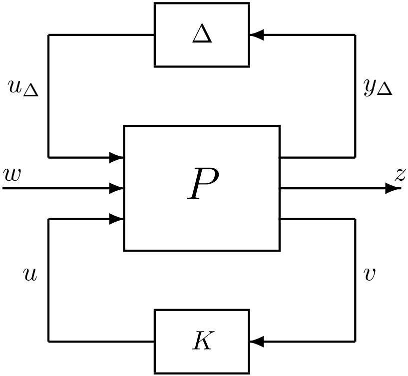


Figure 3: General control configuration (for controller synthesis)

Uncertainty in MIMO Systems

Consider Figure 1 where it is shown how to pull out the perturbation blocks to form Δ and the nominal system N. N is related to P and K by a lower LFT

$$N = F_l(P, K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{6.2}$$

Similarly, the uncertain closed-loop transfer function from w to z, $z = Fw$, is related to N and Δ by an upper LFT,

$$F = F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{6.3}$$

To analyze robust stability of F, we can then arrange the system into the $M\Delta$-structure of Figure 4, where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.

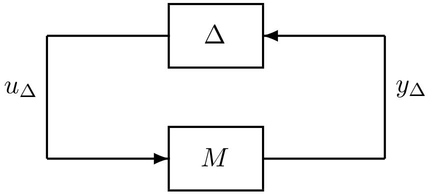


Figure 4: $M\Delta$-structure for robust stability analysis

Uncertainty in MIMO Systems

Representing uncertainty [8.2]
As usual, each individual perturbation is assumed to be stable and is normalized,

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall\omega \tag{6.4}$$

For a complex scalar perturbation we have $|\delta_i(j\omega)| \leq 1, \ \forall\omega$, and for a real scalar perturbation $-1 \leq \delta_i \leq 1$. Since the maximum singular value of a block diagonal matrix is equal to the largest of the maximum singular values of the individual blocks, it then follows for $\Delta = \mathrm{diag}\{\Delta_i\}$ that

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall\omega, \ \forall i \quad \Leftrightarrow \quad \boxed{\|\Delta\|_\infty \leq 1} \tag{6.5}$$

Note that Δ has *structure*, and therefore in the robustness analysis we do *not* want to allow all Δ s.t. (6.5) is satisfied.

Uncertainty in MIMO Systems

Parametric uncertainty:

The representation of parametric uncertainty discussed for SISO systems carries straightforward over to MIMO systems.

Uncertainty in MIMO Systems

Unstructured uncertainty:

We define *unstructured* uncertainty as the use of a “full” complex perturbation matrix Δ, usually with dimensions compatible with those of the plant, where at each frequency any $\Delta(j\omega)$ satisfying $\bar{\sigma}(\Delta(j\omega)) \leq 1$ is allowed.

Six common forms of unstructured uncertainty are shown in Figure 5. In Figure 5(a), (b) and (c) are shown three *feedforward* forms; additive uncertainty, multiplicative input uncertainty and multiplicative output uncertainty:

$$\Pi_A : \qquad G_p = G + E_A; \qquad E_a = w_A \Delta_a \tag{6.6}$$

$$\Pi_I : \qquad G_p = G(I + E_I); \qquad E_I = w_I \Delta_I \tag{6.7}$$

$$\Pi_O : \qquad G_p = (I + E_O)G; \qquad E_O = w_O \Delta_O \tag{6.8}$$

Uncertainty in MIMO Systems

In Figure 5(d), (e) and (f) are shown three *feedback* or *inverse* forms; inverse additive uncertainty, inverse multiplicative input uncertainty and inverse multiplicative output uncertainty:

$$\Pi_{iA}: \qquad G_p = G(I - E_{iA}G)^{-1}; \qquad E_{iA} = w_{iA}\Delta_{iA} \tag{6.9}$$
$$\Pi_{iI}: \qquad G_p = G(I - E_{iI})^{-1}; \qquad E_{iI} = w_{iI}\Delta_{iI} \tag{6.10}$$
$$\Pi_{iO}: \qquad G_p = (I - E_{iO})^{-1}G; \qquad E_{iO} = w_{iO}\Delta_{iO} \tag{6.11}$$

The negative sign in front of the E's does not really matter here since we assume that Δ can have any sign. Δ denotes the normalized perturbation and E the "actual" perturbation. We have here used scalar weights w, so $E = w\Delta = \Delta w$, but sometimes one may want to use matrix weights, $E = W_2 \Delta W_1$ where W_1 and W_2 are given transfer function matrices.

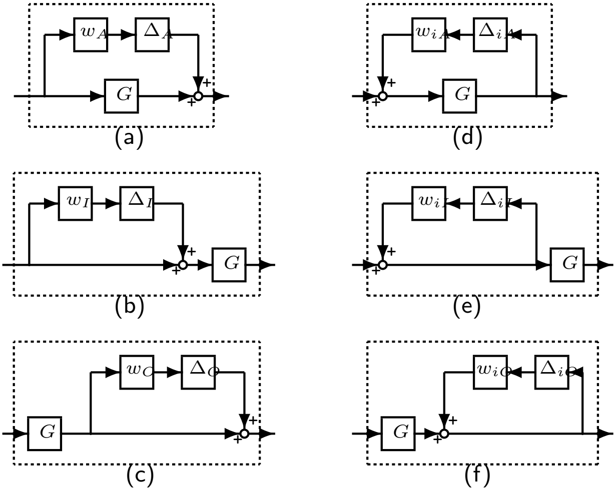


Figure 5: (a) Additive uncertainty, (b) Multiplicative input uncertainty, (c) Multiplicative output uncertainty, (d) Inverse additive uncertainty, (e) Inverse multiplicative input uncertainty, (f) Inverse multiplicative output uncertainty

Obtaining P, N and M [8.3]

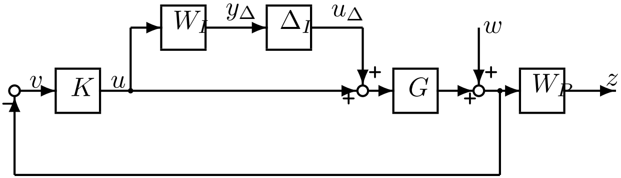


Figure 6: System with multiplicative input uncertainty and performance measured at the output

Example 1: System with input uncertainty (Figure 6). We want to derive the generalized plant P in Figure 3 which has inputs $\begin{bmatrix} u_\Delta & w & u \end{bmatrix}^T$ and outputs $\begin{bmatrix} y_\Delta & z & v \end{bmatrix}^T$. By writing down the equations or simply by inspecting Figure 6 (remember to break the loop before and after both K and Δ) we get

$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix} \tag{6.12}$$

Next, we want to derive the matrix N corresponding to Figure 2. First, partition P to be compatible with K, i.e.

$$P_{11} = \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix}, \; P_{12} = \begin{bmatrix} W_I \\ W_P G \end{bmatrix} \tag{6.13}$$

$$P_{21} = \begin{bmatrix} -G & -I \end{bmatrix}, \; P_{22} = -G \tag{6.14}$$

Uncertainty in MIMO Systems

And then find $N = F_l(P, K)$ using (6.2). We get

$$N = \begin{bmatrix} -W_I K G (I + KG)^{-1} & -W_I K (I + GK)^{-1} \\ W_P G (I + KG)^{-1} & W_P (I + GK)^{-1} \end{bmatrix} \tag{6.15}$$

Alternatively, we can derive N directly from Figure 6 by evaluating the closed-loop transfer function from inputs $\begin{bmatrix} u_\Delta \\ w \end{bmatrix}$ to outputs $\begin{bmatrix} y_\Delta \\ z \end{bmatrix}$ (*without* breaking the loop before and after K).

For example, to derive N_{12}, which is the transfer function from w to y_Δ, we start at the output (y_Δ) and move backwards to the input (w) using the MIMO Rule (we first meet W_I, then $-K$ and we then exit the feedback loop and get the term $(I + GK)^{-1}$).

The upper left block, N_{11}, in (6.15) is the transfer function from u_Δ to y_Δ. This is the transfer function M needed in Figure 4 for evaluating robust stability. Thus, we have $M = -W_I K G (I + KG)^{-1} = -W_I T_I$.

Robust stability & performance [8.4]

1. *Robust stability (RS) analysis*: with a given controller K we determine whether the system remains stable for all plants in the uncertainty set.
2. *Robust performance (RP) analysis*: if RS is satisfied, we determine how "large" the transfer function from exogenous inputs w to outputs z may be for all plants in the uncertainty set.

In Figure 2, w represents the exogenous inputs (normalized disturbances and references), and z the exogenous outputs (normalized errors). We have $z = F(\Delta)w$, where from (6.3)

$$F = F_u(N, \Delta) \stackrel{\Delta}{=} N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{6.16}$$

We here use the $\mathcal{H}_\infty$ norm to define performance and require for RP that $\|F(\Delta)\|_\infty \leq 1$ for all allowed Δ's. A typical choice is $F = w_P S_p$ (the weighted sensitivity function), where w_P is the performance weight (capital P for performance) and S_p represents the set of perturbed sensitivity functions (lower-case p for perturbed).

Uncertainty in MIMO Systems

In terms of the $N\Delta$-structure in Figure 2 our requirements for stability and performance are

$$\text{NS} \quad \overset{\text{def}}{\Leftrightarrow} \quad N \text{ is internally stable} \tag{6.17}$$

$$\text{NP} \quad \overset{\text{def}}{\Leftrightarrow} \quad \|N_{22}\|_\infty < 1; \quad \text{and NS} \tag{6.18}$$

$$\text{RS} \quad \overset{\text{def}}{\Leftrightarrow} \quad F = F_u(N, \Delta) \text{ is stable } \forall\Delta, \|\Delta\|_\infty \leq 1; \text{ and NS} \tag{6.19}$$

$$\text{RP} \quad \overset{\text{def}}{\Leftrightarrow} \quad \|F\|_\infty < 1, \quad \forall\Delta, \|\Delta\|_\infty \leq 1; \text{ and NS} \tag{6.20}$$

Remark 1 Allowed perturbations. For simplicity below we will use the shorthand notation

$$\forall\Delta \quad \text{and} \quad \max_{\Delta} \tag{6.21}$$

to mean "for all Δ's in the set of allowed perturbations", and "maximizing over all Δ's in the set of allowed perturbations". By *allowed perturbations* we mean that the $\mathcal{H}_\infty$ norm of Δ is less or equal to 1, $\|\Delta\|_\infty \leq 1$, and that Δ has a specified block-diagonal structure.

Uncertainty in MIMO Systems

Robust stability of the $M\Delta$-structure [8.5]

Consider the uncertain $N\Delta$-system in Figure 2 for which the transfer function from w to z is, as in (6.16), given by

$$F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{6.22}$$

Suppose that the system is nominally stable (with $\Delta = 0$), that is, N is stable (which means that the whole of N, and not only N_{22}, must be stable). We also assume that Δ is stable. We can see from (6.22) that the only potential source of instability is the feedback term $(I - N_{11}\Delta)^{-1}$. Thus, when we have nominal stability (NS), the stability of the system in Figure 2 is equivalent to the stability of the $M\Delta$-structure in Figure 4 where $M = N_{11}$.

We need now to derive conditions for checking the stability of the $M\Delta$-structure.

Theorem

Determinant stability condition (real or complex perturbations). *Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Consider the convex set of perturbations Δ, such that if Δ' is an allowed perturbation then so is $c\Delta'$ where c is any real scalar such that $|c| \leq 1$. Then the $M\Delta$-system in Figure 4 is stable for all allowed perturbations (we have RS) if and only if*

$$\text{Nyquist plot of } \det\left(I - M(s)\Delta(s)\right) \text{ does not encircle the origin, } \forall\Delta \tag{6.23}$$

$$\Leftrightarrow \quad \boxed{\det\left(I - M(j\omega)\Delta(j\omega)\right) \neq 0, \quad \forall\omega, \forall\Delta} \tag{6.24}$$

$$\Leftrightarrow \quad \lambda_i(M\Delta) \neq 1, \quad \forall i, \forall\omega, \forall\Delta \tag{6.25}$$

Note that (6.23) is simply the generalization of the Nyquist theorem.

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Theorem: Generalized (MIMO) Nyquist theorem. Let P_{ol} denote the number of open-loop unstable poles in $L(s)$. The closed-loop system with loop transfer function $L(s)$ and negative feedback is stable if and only if the Nyquist plot of $\det(I + L(s))$
i) makes P_{ol} anti-clockwise encirclements of the origin, and
ii) does not pass through the origin.

Note
By "Nyquist plot of $\det(I + L(s))$" we mean "the image of $\det(I + L(s))$ as s goes clockwise around the Nyquist D-contour".

RS for unstructured uncertainty [8.6]

Theorem

RS for unstructured ("full") perturbations. *Assume that the nominal system $M(s)$ is stable (NS) and that the perturbation $\Delta(s)$ are stable ($\Delta(s)$ is allowed to be any (full) complex transfer function matrix). Then the $M\Delta$-system in Figure 4 is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \leq 1$ (i.e. we have RS) if and only if*

$$\boxed{\bar{\sigma}(M(j\omega)) < 1 \quad \forall w} \qquad \Leftrightarrow \qquad \boxed{\|M\|_\infty < 1} \tag{6.26}$$

Remark 1: Condition (6.26) may be rewritten as

$$RS \qquad \Leftrightarrow \quad \bar{\sigma}(M(j\omega))\bar{\sigma}(\Delta(j\omega)) < 1 \quad \forall w, \forall \Delta \tag{6.27}$$

The sufficiency of (6.27) ($\Leftarrow$) follows directly from the small-gain theorem by choosing $L = M\Delta$. The small-gain theorem applies to any operator norm satisfying $\|AB\| \leq \|A\|\|B\|$.
Remark 2: The $\mathcal{H}_\infty$ is one of such operators. This implies that the RS condition (6.27) is both necessary and sufficient.

Uncertainty in MIMO Systems

Small Gain Theorem. Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if

$$\|L(j\omega)\| < 1 \quad \forall\omega \tag{6.28}$$

where $\|L\|$ denotes any matrix norm satisfying $\|AB\| \leq \|A\| \cdot \|B\|$, for example the singular value $\bar{\sigma}(L)$.

Uncertainty in MIMO Systems

Application of the unstructured RS-condition [8.6.1]:

We will now present necessary and sufficient conditions for robust stability (RS) for each of the six single unstructured perturbations in Figure 5 with

$$E = W_2 \Delta W_1, \quad \|\Delta\|_\infty \leq 1 \tag{6.29}$$

To derive the matrix M we simply "isolate" the perturbation, and determine the transfer function matrix

$$M = W_1 M_0 W_2 \tag{6.30}$$

from the output to the input of the perturbation, where M_0 for each of the six cases becomes (disregarding some negative signs which do not affect the subsequent robustness condition) is given by

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$$G_p = G + E_A: \quad M_0 = K(I+GK)^{-1} = KS \tag{6.31}$$

$$G_p = G(I + E_I): \quad M_0 = K(I+GK)^{-1}G = T_I \tag{6.32}$$

$$G_p = (I + E_O)G: \quad M_0 = GK(I+GK)^{-1} = T \tag{6.33}$$

$$G_p = G(I - E_{iA}G)^{-1}: \quad M_0 = (I+GK)^{-1}G = SG \tag{6.34}$$

$$G_p = G(I - E_{iI})^{-1}: \quad M_0 = (I+KG)^{-1} = S_I \tag{6.35}$$

$$G_p = (I - E_{iO})^{-1}G: \quad M_0 = (I+GK)^{-1} = S \tag{6.36}$$

Theorem 2 then yields

$$\mathrm{RS} \quad \Leftrightarrow \quad \|W_1 M_0 W_2(j\omega)\|_\infty < 1, \forall\ w \tag{6.37}$$

For instance, from second equation (6.32) and (6.37) we get for multiplicative input uncertainty with a scalar weight:

$$\mathrm{RS}\ \forall G_p = G(I + w_I \Delta_I),\ \|\Delta_I\|_\infty \le 1 \ \Leftrightarrow\ \|w_I T_I\|_\infty < 1 \tag{6.38}$$

Note that the SISO-condition (5.15)

$$\boxed{\mathrm{RS} = |T| < 1/|w_I|, \quad \forall\omega} \tag{6.39}$$

follows as a special case of (6.38). Similarly, (5.20)

$$\boxed{\mathrm{RS} \quad \Leftrightarrow \quad |S| < 1/|w_{oI}|, \quad \forall\omega} \tag{6.40}$$

follows as a special case of the inverse multiplicative output uncertainty in (6.36):

$$\begin{aligned} &\mathrm{RS}\ \forall G_p = (I - w_{iO}\Delta_{iO})^{-1}G, \\ &\|\Delta_{iO}\|_\infty \leq 1 \Leftrightarrow \|w_{iO}S\|_\infty < 1 \end{aligned} \tag{6.41}$$

In general, the unstructured uncertainty descriptions in terms of a single perturbation are not "tight" (in the sense that at each frequency all complex perturbations satisfying $\bar{\sigma}(j\omega) \leq 1$ may not occur in practice). Thus, the above RS conditions are often conservative. In order to get tighter conditions we must use a tighter uncertainty description in terms of a block diagonal Δ.

RS with structured uncertainty [8.7]

Consider now the presence of structured uncertainty, where $\Delta = \mathrm{diag}\{\Delta_i\}$ is block-diagonal. To test for robust stability we rearrange the system into the $M\Delta$-structure and we have from (6.26)

$$\mathrm{RS} \quad \mathrm{if} \quad \bar{\sigma}(M(j\omega)) < 1, \forall\omega \tag{6.42}$$

We have here written "if" rather than "if and only if" since this condition is only necessary for RS when Δ has "no structure" (full-block uncertainty).

Can we take advantage of the fact that $\Delta = \mathrm{diag}\{\Delta_i\}$ has structure to obtain tighter RS conditions? To reduce conservatism, we introduce the block-diagonal scaling matrix

$$D = \mathrm{diag}\{d_i I_i\} \tag{6.43}$$

where d_i is a scalar and I_i is an identity matrix of the same dimension as the i'th perturbation block, Δ_i (Figure 7). This clearly has no effect on stability.

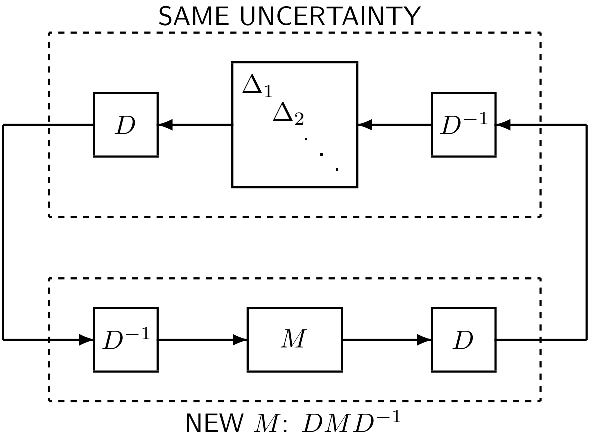

Figure 7: Use of block-diagonal scalings, $\Delta D = D\Delta$

Note that with the chosen form for the scalings, we have for each perturbation block $\Delta_i = d_i \Delta_i d_i^{-1}$; that is, we have $\Delta = D\Delta D^{-1} \Rightarrow$ Same uncertainty!

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$$\text{RS} \quad \text{if} \quad \bar{\sigma}(DMD^{-1}) < 1, \forall \omega \tag{6.44}$$

This applies for any D in (6.43), and therefore the "most improved" (least conservative) RS-condition is obtained by minimizing at each frequency the scaled singular value, and we have

$$\boxed{\text{RS} \quad \text{if} \quad \min_{D(\omega)\in\mathcal{D}} \bar{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1}) < 1, \forall \omega} \tag{6.45}$$

where $\mathcal{D}$ is the set of block-diagonal matrices whose structure is compatible to that of Δ, i.e, $\Delta D = D\Delta$. Note that when Δ is a full (unstructured) matrix, we must select $D = dI$ and we have $\bar{\sigma}(DMD^{-1}) = \bar{\sigma}(M)$ (RS conditions for unstructured uncertainty). However, when Δ has structure, we get more degrees of freedom in D and $\bar{\sigma}(DMD^{-1})$ may be significantly smaller than $\bar{\sigma}(M)$.

This motivates the introduction of the structure singular value, $\mu(M)$ satisfying $\mu(M) \leq \min_D \bar{\sigma}(DMD^{-1})$.

The structured singular value [8.8]

The structured singular value (denoted Mu, mu, SSV or μ) is a function which provides a generalization of the singular value, $\bar{\sigma}$, and the spectral radius, ρ. We will use μ to get necessary and sufficient conditions for robust stability and also for robust performance. How is μ defined? A simple statement is:

> *Find the smallest structured* Δ *(measured in terms of* $\bar{\sigma}(\Delta)$*) which makes* $\det(I - M\Delta) = 0$*; then* $\mu(M) = 1/\bar{\sigma}(\Delta)$*.*

Mathematically,

$$\mu(M)^{-1} \overset{\Delta}{=} \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for structured } \Delta\} \quad (6.46)$$

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Clearly, $\mu(M)$ depends not only on M but also on the allowed structure for Δ. This is sometimes shown explicitly by using the notation $\mu_\Delta(M)$.

Remark. For the case where Δ is "unstructured" (a full matrix), the smallest Δ which yields singularity has $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M)$, and we have $\mu(M) = \bar{\sigma}(M)$.

Remark. As one might guess, we have that $\mu(M) \leq \min_D \bar{\sigma}(DMD^{-1})$. In fact, for block-diagonal complex perturbations we generally have that $\mu(M)$ is very close to $\min_D \bar{\sigma}(DMD^{-1})$.

Example

Full perturbation *(Δ is unstructured). Consider*

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3.162 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}^H \tag{6.47}$$

The perturbation

$$\Delta = \frac{1}{\sigma_1} v_1 u_1^H = \frac{1}{3.162} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \begin{bmatrix} 0.894 & -0.447 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.200 & 0.200 \\ -0.100 & -0.100 \end{bmatrix} \tag{6.48}$$

with $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M) = 1/3.162 = 0.316$ *makes* $\det(I - M\Delta) = 0$. *Thus* $\mu(M) = 3.162$ *when* Δ *is a full matrix.*

Note that the perturbation Δ in (6.48) is a full matrix. If we restrict Δ to be diagonal then we need a larger perturbation to make $\det(I - M\Delta) = 0$. This is illustrated next.

Example

Continued. Diagonal perturbation *(Δ is structured). For the matrix M in (6.47), the smallest diagonal Δ which makes $\det(I - M\Delta) = 0$ is*

$$\Delta = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.49}$$

with $\bar{\sigma}(\Delta) = 0.333$. Thus $\mu(M) = 3$ when Δ is a diagonal matrix.

Definition

Structured Singular Value. Let M be a given complex matrix and let $\Delta = \mathrm{diag}\{\Delta_i\}$ denote a set of complex matrices with $\bar{\sigma}(\Delta) \leq 1$ and with a given block-diagonal structure (in which some of the blocks may be repeated and some may be restricted to be real). The real non-negative function $\mu(M)$, called the structured singular value, is defined by

$$\mu(M) \triangleq \frac{1}{\min\{k_m | \det(I - k_m M\Delta) = 0, \bar{\sigma}(\Delta) \leq 1\}} \tag{6.50}$$

If no such structured Δ exists then $\mu(M) = 0$.

A value of $\mu = 1$ means that there exists a perturbation with $\bar{\sigma}(\Delta) = 1$ which is just large enough to make $I - M\Delta$ singular. A larger value of μ is “bad” as it means that a smaller perturbation makes $I - M\Delta$ singular, whereas a smaller value of μ is “good”.

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Properties of the structured singular value:

Check Sections 8.8.1, 8.8.2 and 8.8.3 in the book.

RS - Structured uncertainty [8.9]

Consider stability of the $M\Delta$-structure in Figure 4 for the case where Δ is a set of norm-bounded block-diagonal perturbations. From the determinant stability condition:

$$RS \Leftrightarrow \det\left(I - M(j\omega)\Delta(j\omega)\right) \neq 0, \quad \forall\omega, \forall\Delta, \bar{\sigma}(\Delta(j\omega)) \leq 1 \quad \forall\omega \quad (6.51)$$

This is just a "yes/no" condition. To find the factor k_m by which the system is robustly stable, we scale the uncertainty Δ by k_m, and look for the smallest k_m that yields "borderline instability," namely

$$\det\left(I - k_m M(j\omega)\Delta(j\omega)\right) = 0 \quad (6.52)$$

By definition, this value is $k_m = 1/\mu(M)$. We obtain the following necessary and sufficient condition for stability.

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Theorem

RS for block-diagonal perturbations (real or complex). *Assume that the nominal system M and the perturbations Δ are stable. Then the $M\Delta$-system in Figure 4 is stable for all allowed perturbations with $\bar{\sigma}(\Delta) \leq 1, \forall\omega$, if and only if*

$$\boxed{\mu(M(j\omega)) < 1, \quad \forall\omega} \tag{6.53}$$

Condition (6.53) for robust stability may be rewritten as

$$\mathrm{RS} \;\Leftrightarrow\; \mu(M(j\omega))\,\bar{\sigma}(\Delta(j\omega)) < 1, \quad \forall\omega \tag{6.54}$$

which may be interpreted as a "generalized small gain theorem" that also takes into account the *structure* of Δ.

Example: RS with diagonal input uncertainty

Consider robust stability of the feedback system in Figure 6 for the case when the multiplicative input uncertainty is diagonal. A nominal 2×2 plant and the controller (which represents PI-control of a distillation process using the DV-configuration) is given by

$$G(s) = \tfrac{1}{\tau s+1} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}; \quad K(s) = \tfrac{1+\tau s}{s} \begin{bmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{bmatrix} \tag{6.55}$$

(time in minutes). The controller results in a nominally stable system with acceptable performance. Assume there is complex multiplicative uncertainty in *each* manipulated input of magnitude

$$w_I(s) = \frac{s+0.2}{0.5s+1} \tag{6.56}$$

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On rearranging the block diagram to match the $M\Delta$-structure in Figure 4 we get $M = w_I KG(I + KG)^{-1} = w_I T_I$ (recall (6.15)), and the RS-condition $\mu(M) < 1$ in Theorem 4 yields

$$\mathrm{RS} \Leftrightarrow \mu_{\Delta_I}(T_I) < \frac{1}{|w_I(j\omega)|} \quad \forall\omega, \quad \Delta_I = \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix} \tag{6.57}$$

This condition is shown graphically in Figure 8 so the system is robustly stable. Also in Figure 8, $\bar{\sigma}(T_I)$ can be seen to be larger than $1/|w_I(j\omega)|$ over a wide frequency range. This shows that the system would be unstable for full-block input uncertainty (Δ_I full).

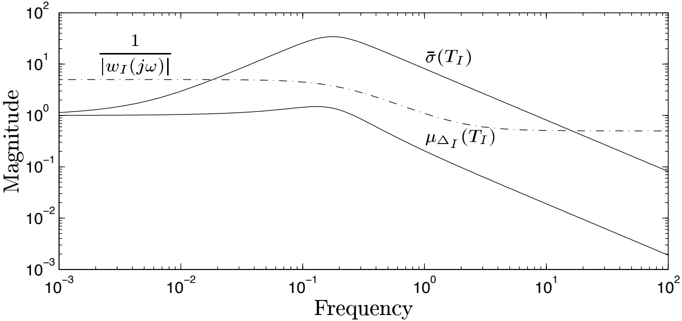

Figure 8: Robust stability for diagonal input uncertainty is guaranteed since $\mu_{\Delta_I}(T_I) < 1/|w_I|, \ \forall\omega$. The use of unstructured uncertainty and $\bar{\sigma}(T_I)$ is conservative.

Example: RS of Spinning Satellite [3.7.1]

Angular velocity control of a satellite spinning about one of its principal axes:

$$G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}; \quad a = 10 \tag{6.58}$$

A minimal, state-space realization, $G = C(sI - A)^{-1}B + D$, is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right] \tag{6.59}$$

Poles at $s = \pm ja$ For stabilization:

$$K = I$$

$$T(s) = GK(I + GK)^{-1} = \frac{1}{s+1} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \tag{6.60}$$

Nominal stability (NS). Two closed loop poles at $s = -1$ and

$$A_{cl} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nominal performance (NP).

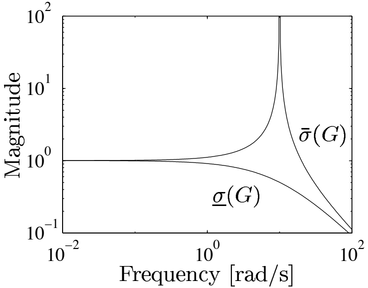

Figure 9: Singular values - Spinning satellite (6.58)

- $\underline{\sigma}(L) \leq 1 \quad \forall \omega$ poor performance in low gain direction
- g_{12}, g_{21} large $\Rightarrow$ strong interaction

Robust stability (RS).

Check stability: one loop at a time.

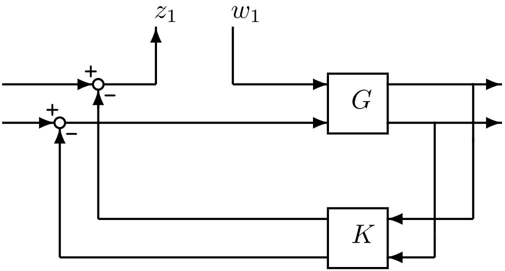

Figure 10: Checking stability margins "one-loop-at-a-time"

$$\frac{z_1}{w_1} \triangleq L_1(s) = \frac{1}{s} \Rightarrow GM = \infty, PM = 90^\circ \tag{6.61}$$

Uncertainty in MIMO Systems

- Good Robustness? NO
- Consider perturbation in each feedback channel

$$u_1' = (1+\epsilon_1)u_1, \quad u_2' = (1+\epsilon_2)u_2 \tag{6.62}$$

$$B' = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$$

Closed-loop state matrix:

$$A'_{cl} = A - B'KC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

Uncertainty in MIMO Systems

Characteristic polynomial:

$$\det(sI - A'_{cl}) = s^2 + \underbrace{(2 + \epsilon_1 + \epsilon_2)}_{a_1} s + \\ + \underbrace{1 + \epsilon_1 + \epsilon_2 + (a^2 + 1)\epsilon_1 \epsilon_2}_{a_0}$$

Stability for $(-1 < \epsilon_1 < \infty, \epsilon_2 = 0)$ and $(\epsilon_1 = 0, -1 < \epsilon_2 < \infty)$ (GM=∞)

But only *small simultaneous changes* in the two channels: for example, let $\epsilon_1 = -\epsilon_2$, then the system is unstable $(a_0 < 0)$ for

$$|\epsilon_1| > \frac{1}{\sqrt{a^2 + 1}} \approx 0.1$$

Summary. Checking single-loop margins is inadequate for MIMO problems.

- $\mu(T) = \bar{\sigma}(T)$ irrespective of the structure of the COMPLEX multiplicative uncertainty perturbation (full block, diagonal, repeated scalar).
- We can tolerate more than 100% uncertainty above 10 rad/sec. At low frequencies we have $\mu(T) \sim 10$, so to guarantee RS we need less than 10% uncertainty. This confirms previous results showing that real perturbations $\delta_1 = 0.1$ and $\delta_2 = -0.1$ yield instability. Then, use of complex rather than real perturbations is not conservative in this case, at least for diagonal uncertainty.
- For repeated scalar uncertainty, there is a difference between real and complex uncertainties. The characteristic polynomial shows that $\delta_1 = \delta_2 = -1$ (100% uncertainty) and $\delta_1 = \delta_2 = j0.1$ (10% uncertainty) yield instability.

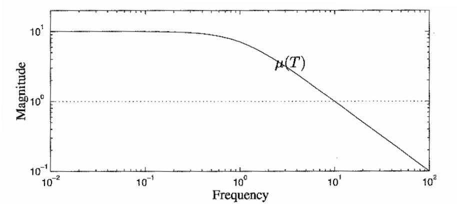

Figure 11: μ-plot for spinning satellite

Uncertainty in MIMO Systems

Robust performance [8.10]

- With an $\mathcal{H}_\infty$ performance objective, the RP-condition is identical to a RS-condition with an additional perturbation block as shown in Figure 12.
- Step B in Figure 12 is the key step.
- Δ_P (where capital P denotes Performance) is always a full matrix. It is a fictitious uncertainty block representing the $\mathcal{H}_\infty$ performance specification.

Testing RP using μ [8.10.1]:

Theorem

Robust performance. *Rearrange the uncertain system into the $N\Delta$-structure of Figure 12. Assume nominal stability such that N is (internally) stable. Then*

$$\begin{aligned} \mathrm{RP} \quad &\overset{\text{def}}{\Leftrightarrow} \quad \|F\|_\infty = \|F_u(N,\Delta)\|_\infty < 1, \quad \forall \|\Delta\|_\infty \le 1 \\ &= \quad \boxed{\mu_{\widehat{\Delta}}(N(j\omega)) < 1, \quad \forall w} \end{aligned} \tag{6.63}$$

where μ is computed with respect to the structure

$$\widehat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \tag{6.64}$$

and Δ_P is a full complex perturbation with the same dimensions as F^T.

NOTE: Recall that stability of the $M\Delta$ structure where Δ is a full complex matrix is equivalent to $\|M\|_\infty < 1$. From this theorem, the RP condition $\|F\|_\infty < 1$ is equivalent to the RS of the $F\Delta_P$ structure where Δ_P is a full complex matrix.

Uncertainty in MIMO Systems

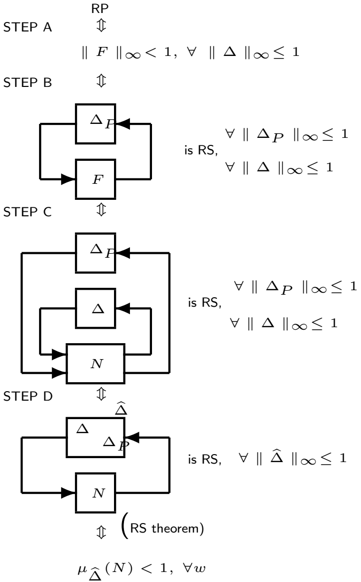

Figure 12: RP as a special case of structured RS.

Summary of μ-conditions for NP, RS, RP [8.10.2]:
Rearrange the uncertain system into the $N\Delta$- structure, where the block-diagonal perturbations satisfy $\|\Delta\|_\infty \leq 1$.
Introduce

$$F = F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

and let the performance requirement (RP) be $\|F\|_\infty \leq 1$ for all allowable perturbations. Then we have:

$$\begin{aligned}
\mathrm{NS} \quad &= \quad N \text{ (internally) stable} && (6.65)\\
\mathrm{NP} \quad &= \quad \bar{\sigma}(N_{22}) = \mu_{\Delta_P}(N_{22}) < 1,\ \forall\omega,\ \text{and NS} && (6.66)\\
\mathrm{RS} \quad &= \quad \mu_{\Delta}(N_{11}) < 1,\ \forall\omega,\ \text{and NS} && (6.67)\\
\mathrm{RP} \quad &= \quad \mu_{\widetilde{\Delta}}(N) < 1,\ \forall\omega,\ \widetilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}, && (6.68)\\
&\qquad \text{and NS}
\end{aligned}$$

Here Δ is a block-diagonal matrix (its detailed structure depends on the uncertainty we are representing), whereas Δ_P always is a full complex matrix. Note that nominal stability (NS) must be tested separately in all cases.

Application: RP - input uncertainty [8.11]:

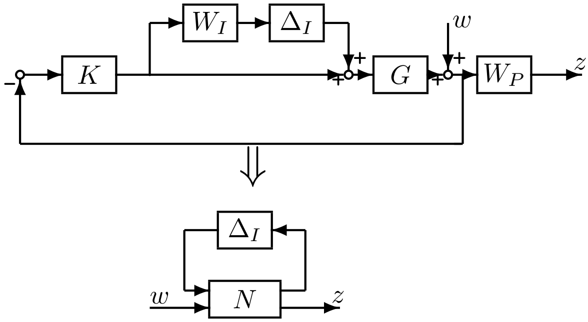

Figure 13: Robust performance of system with input uncertainty

$$RP \Leftrightarrow \|w_P(I + G_P K)^{-1}\|_\infty < 1, \quad \forall G_p \tag{6.69}$$

$$G_P = G(I + w_I \Delta_I), \quad \|\Delta_I\|_\infty \leq 1 \tag{6.70}$$

Uncertainty in MIMO Systems

Interconnection matrix [8.11.1]:

On rearranging the system into the $N\Delta$-structure, as shown in Figure 13, we get

$$N = \begin{bmatrix} w_I T_I & w_I KS \\ w_P SG & w_P S \end{bmatrix} \tag{6.71}$$

where $T_I = KG(I + KG)^{-1}$, $S = (I + GK)^{-1}$ and for simplicity we have omitted the negative signs in the 1,1 and 1,2 blocks of N, since $\mu(N) = \mu(UN)$ with unitary $U = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$.

For a given controller K we can now test for NS, NP, RS and RP using (6.65)-(6.68). Here $\Delta = \Delta_I$ may be a full or diagonal matrix (depending on the physical situation).

Uncertainty in MIMO Systems

RP with input uncertainty for SISO system [8.11.2]:

For a SISO system, conditions (6.65)-(6.68) with N as in (6.71) become

$$\begin{aligned} \mathrm{NS} &= S,\ SG,\ KS \text{ and } T_I \text{ are stable} && (6.72) \\ \mathrm{NP} &= |w_P S| < 1,\ \forall\omega && (6.73) \\ \mathrm{RS} &= |w_I T_I| < 1,\ \forall\omega && (6.74) \\ \mathrm{RP} &= |w_P S| + |w_I T_I| < 1,\ \forall\omega && (6.75) \end{aligned}$$

Since Δ_I and Δ_P are scalars in this case, the RP conditions follows from (see Exercise 8.18) the fact that

$$\mu(N) = \mu \begin{bmatrix} w_I T_I & w_I K S \\ w_P S G & w_P S \end{bmatrix} = \mu \begin{bmatrix} w_I T_I & w_I T_I \\ w_P S & w_P S \end{bmatrix} = |w_I T_I| + |w_P S|$$

where $T_I = KSG$. For SISO systems, $T_I = T$.

RP for 2×2 distillation process [8.11.3]:

Consider again the distillation process example from Chapter 3 (Motivating Example No. 2) and the corresponding inverse-based controller:

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}; K(s) = \frac{0.7}{s} G(s)^{-1} \tag{6.76}$$

The controller provides a nominally decoupled system with

$$L = lI, \;\; S = \epsilon I \text{ and } T = tI \tag{6.77}$$

where

$$l = \frac{0.7}{s}, \epsilon = \frac{1}{1+l} = \frac{s}{s+0.7},$$
$$t = 1 - \epsilon = \frac{0.7}{s+0.7} = \frac{1}{1.43s+1}$$

We have used ϵ for the nominal sensitivity in each loop to distinguish it from the Laplace variable s.

Uncertainty in MIMO Systems

Weights for uncertainty and performance:

$$w_I(s) = \frac{s + 0.2}{0.5s + 1}; \quad w_P(s) = \frac{s/2 + 0.05}{s} \tag{6.78}$$

The weight $w_I(s)$ may approximately represent a 20% gain error and a neglected time delay of 0.9 min (see Section 7.4.5). $|w_I(j\omega)|$ levels off at 2 (200% uncertainty) at high frequencies. The performance weight $w_P(s)$ specifies integral action, a closed-loop bandwidth of about 0.05 [rad/min] (which is relatively slow in the presence of an allowed time delay of 0.9 min) and a maximum peak for $\bar{\sigma}(S)$ of $M_s = 2$.

Uncertainty in MIMO Systems

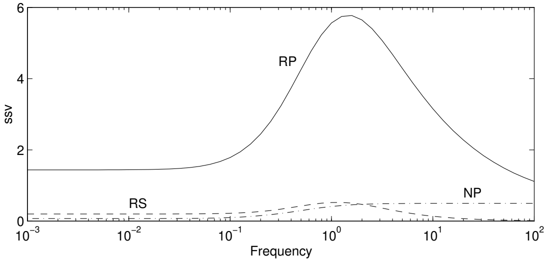

Figure 14: μ-plots for distillation process with decoupling controller

NS Yes.

NP With the decoupling controller we have

$$\bar{\sigma}(N_{22}) = \bar{\sigma}(w_P S) = \left| \frac{s/2 + 0.05}{s + 0.7} \right|$$

(dashed-dot line in Figure 14 $\Leftarrow$ NP is OK.)

RS Since in this case $w_I T_I = w_I T$ is a scalar times the identity matrix, we have, independent of the structure of Δ_I, that

$$\mu_{\Delta_I}(w_I T_I) = |w_I t| = \left| 0.2 \frac{5s + 1}{(0.5s + 1)(1.43s + 1)} \right|$$

and we see from the dashed line in Figure 14 that RS is OK.

RP Poor.

Table 1: **MATLAB program for μ-analysis**

```
% Uses the Mu toolbox
G0 = [87.8 -86.4; 108.2 -109.6];
dyn = nd2sys(1,[75 1]);
Dyn=daug(dyn,dyn); G=mmult(Dyn,G0);
%
% Inverse-based control.
%
dynk=nd2sys([75 1],[1 1.e-5],0.7);
Dynk=daug(dynk,dynk); Kinv=mmult(Dynk,minv(G0));
%
% Weights.
%
wp=nd2sys([10 1],[10 1.e-5],0.5); Wp=daug(wp,wp);
wi=nd2sys([1 0.2],[0.5 1]); Wi=daug(wi,wi);
%
```

Table 2: **MATLAB program for μ-analysis**

```
% Generalized plant P.
systemnames = 'G Wp Wi';
inputvar = '[ydel(2); w(2) ; u(2)]';
outputvar = '[Wi; Wp; -G-w]';
input_to_G = '[u+ydel]';
input_to_Wp = '[G+w]'; input_to_Wi = '[u]';
sysoutname = 'P';
cleanupsysic = 'yes'; sysic;
%
N = starp(P,Kinv); omega = logspace(-3,3,61);
Nf = frsp(N,omega);
%
% mu for RP.
blk = [1 1; 1 1; 2 2];
[mubnds,rowd,sens,rowp,rowg] = mu(Nf,blk,'c');
muRP = sel(mubnds,':',1); pkvnorm(muRP)           % (ans = 5.7726).
```

Table 3: **MATLAB program for μ-analysis**

```
% Worst-case weighted sensitivity
%
[delworst,muslow,musup] = wcperf(Nf,blk,1); musup       % (musup = 44.93 for
%                                                        %    delta=1).
% mu for RS.
%
Nrs=sel(Nf,1:2,1:2);
[mubnds,rowd,sens,rowp,rowg]=mu(Nrs,[1 1; 1 1],'c');
muRS = sel(mubnds,':',1); pkvnorm(muRS)                  % (ans = 0.5242).
%
% mu for NP (= max.  singular value of Nnp).
%
Nnp=sel(Nf,3:4,3:4);
[mubnds,rowd,sens,rowp,rowg]=mu(Nnp,[2 2],'c');
muNP = sel(mubnds,':',1); pkvnorm(muNP)                  % (ans = 0.5000).
vplot('liv,m',muRP,muRS,muNP);
```

μ-synthesis and DK-iteration [8.12]

The structured singular value μ is a very powerful tool for the analysis of robust performance with a given controller. However, one may also seek to find the controller that minimizes a given μ-condition: this is the μ-synthesis problem: $\min_K \mu(N)$

DK-iteration [8.12.1]:

At present there is no direct method to synthesize a μ-optimal controller. However, for complex perturbations a method known as DK-iteration is available. It combines $\mathcal{H}_\infty$-synthesis and μ-analysis, and often yields good results. The starting point is the upper bound on μ in terms of the scaled singular value

$$\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

Uncertainty in MIMO Systems

- The structured singular value μ for *complex* perturbations is bounded by

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \tag{6.79}$$

$$\Delta = \delta I \text{ (δ is a complex scalar): } \mu(M) = \rho(M) \tag{6.80}$$

$$\Delta \text{ is a full complex matrix: } \mu(M) = \bar{\sigma}(M) \tag{6.81}$$

- Consider any matrix D which commutes with Δ: that is $\Delta D = D\Delta$. Then

$$\mu(DM) = \mu(MD) \text{ and } \mu(DMD^{-1}) = \mu(M) \tag{6.82}$$

- **Improved upper bound.** Define $\mathcal{D}$ to be the set of matrices D which commute with Δ (i.e., satisfy $D\Delta = \Delta D$). Then

$$\mu(M) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) \tag{6.83}$$

  - This optimization is convex in D, i.e., has only one global minimum.
  - The inequality is indeed an equality if there are three or fewer blocks in Δ.
  - Numerical evidence suggests that the bound is tight.

The idea is to find the controller that minimizes the peak value over frequency of this upper bound, namely

$$\min_K\left(\min_{D\in\mathcal{D}} \|DN(K)D^{-1}\|_\infty\right) \tag{6.84}$$

by alternating between minimizing $\|DN(K)D^{-1}\|_\infty$ with respect to either K or D (while holding the other fixed).

1. **K-step.** Synthesize an $\mathcal{H}_\infty$ controller for the scaled problem, $\min_K \|DN(K)D^{-1}\|_\infty$ with fixed $D(s)$.
2. **D-step.** Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

Example: μ-synthesis with DK-iteration [8.12.4]:

Simplified distillation process

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{6.85}$$

The uncertainty weight $w_I I$ and performance weight $w_P I$ are given in (6.78), and are shown graphically in Figure 15. The objective is to minimize the peak value of $\mu_{\widetilde{\Delta}}(N)$, where N is given in (6.71) and $\widetilde{\Delta} = \mathrm{diag}\{\Delta_I, \Delta_P\}$. We will consider diagonal input uncertainty (which is always present in any real problem), so Δ_I is a 2×2 diagonal matrix. Δ_P is a full 2×2 matrix representing the performance specification.

Uncertainty in MIMO Systems

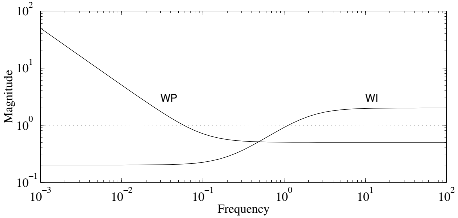

Figure 15: Uncertainty and performance weights. Notice that there is a frequency range ("window") where both weights are less than one in magnitude.

Table 4: **MATLAB program to perform DK-iteration**

```
% Uses the Mu toolbox
G0 = [87.8 -86.4; 108.2 -109.6];
dyn = nd2sys(1,[75 1]); Dyn = daug(dyn,dyn);
G = mmult(Dyn,G0);
%
% Weights.
%
wp = nd2sys([10 1],[10 1.e-5],0.5);             % Approximated
wi = nd2sys([1 0.2],[0.5 1]);                   % integrator.
Wp = daug(wp,wp); Wi = daug(wi,wi);
%
% Generalized plant P. %
systemnames = 'G Wp Wi';
inputvar = '[ydel(2); w(2) ; u(2)]';
outputvar = '[Wi; Wp; -G-w]';
input_to_G = '[u+ydel]';
input_to_Wp = '[G+w]'; input_to_Wi = '[u]';
sysoutname = 'P'; cleanupsysic = 'yes';
sysic;
%
```

Table 5: **MATLAB program to perform DK-iteration**

```
% Initialize.
%
omega = logspace(-3,3,61);
blk = [1 1; 1 1; 2 2];
nmeas=2; nu=2; gmin=0.9; gamma=2; tol=0.01; d0 = 1;
dsysl = daug(d0,d0,eye(2),eye(2)); dsysr=dsysl;
%
% START ITERATION.
%
% STEP 1:  Find H-infinity optimal controller
% with given scalings:
%
DPD = mmult(dsysl,P,minv(dsysr)); gmax=1.05*gamma;
[K,Nsc,gamma] = hinfsyn(DPD,nmeas,nu,gmin,gmax,tol);
Nf=frsp(Nsc,omega);                                       % (Remark:
%                                                         %  Without scaling:
%                                                         %  N=starp(P,K);).
% STEP 2:  Compute mu using upper bound:
%
[mubnds,rowd,sens,rowp,rowg] = mu(Nf,blk,'c');
vplot('liv,m',mubnds); murp=pkvnorm(mubnds,inf)
%
% STEP 3:  Fit resulting D-scales:
%
[dsysl,dsysr]=musynflp(dsysl,rowd,sens,blk,nmeas,nu);    % choose 4th order.
%
% New Version:
% [dsysL,dsysR]=msf(Nf,mubnds,rowd,sens,blk);            % order:  4, 4, 0.
% dsysl=daug(dsysL,eye(2)); dsysr=daug(dsysR,eye(2));
%
% GOTO STEP 1 (unless satisfied with murp).
%
```

Iteration No. 1.

Step 1: With the initial scalings, $D^0 = I$, the $\mathcal{H}_\infty$ software produced a 6 state controller (2 states from the plant model and 2 from each of the weights) with an $\mathcal{H}_\infty$ norm of $\gamma = 1.1823$.

Step 2: The upper μ-bound gave the μ-curve shown as curve "Iter. 1" in Figure 16, corresponding to a peak value of μ=1.1818.

Step 3: The frequency-dependent $d_1(\omega)$ and $d_2(\omega)$ from Step 2 were each fitted using a 4th order transfer function. $d_1(w)$ and the fitted 4th-order transfer function (dotted line) are shown in Figure 17 and labelled "Iter. 1".

Iteration No. 2.

Step 1: With the 8 state scaling $D^1(s)$ the $\mathcal{H}_\infty$ software gave a 22 state controller and $\|D^1 N (D^1)^{-1}\|_\infty = 1.0238$.

Iteration No. 3.

Step 1: With the scalings $D^2(s)$ the $\mathcal{H}_\infty$ norm was only slightly reduced from 1.024 to 1.019.

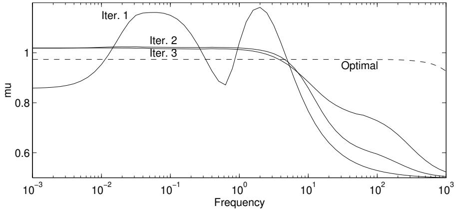

Figure 16: Change in μ during DK-iteration

Uncertainty in MIMO Systems

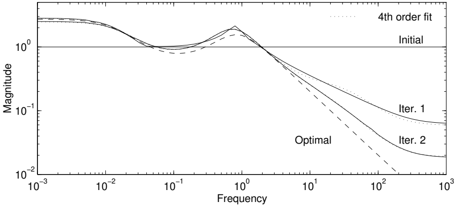

Figure 17: Change in D-scale d_1 during DK-iteration

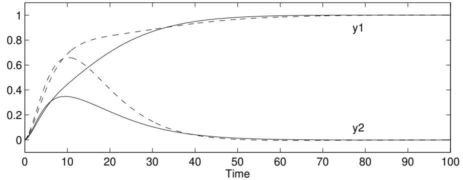

Figure 18: Setpoint response for μ-"optimal" controller K_3. Solid line: nominal plant. Dashed line: uncertain plant G_3'