Multivariable Robust Control

Lecture 5 (Meetings 11-12)

Chapter 7: Uncertainty and Robustness for SISO Systems

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Uncertainty in SISO Systems

Introduction [7.1]

A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller. These differences are referred to as *model/plant mismatch* or simply *model uncertainty*.

Uncertainty in SISO Systems

Our approach is:

1. Determine the uncertainty set: find a mathematical representation of the model uncertainty ("clarify what we know about what we don't know").
2. Check Robust stability (RS): determine whether the system remains stable for all plants in the uncertainty set.
3. Check Robust performance (RP): if RS is satisfied, determine whether the performance specifications are met for all plants in the uncertainty set.

Notation:

Π – a set of possible perturbed plant models ("uncertainty set").

$G(s) \in \Pi$ – nominal plant model (with no uncertainty).

$G_p(s) \in \Pi$ and $G'(s) \in \Pi$ – particular perturbed plant models.

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Representing uncertainty [7.2]

Sources of uncertainty

1. Unknown parameters in the model
2. Unmodeled nonlinearities in the linear model
3. At high frequency both structure+order of model are unknown
4. Detailed model $\rightarrow$ low-order simpler model for control synthesis
5. Synthesized controller may be different from implemented controller
   - Implementation of controller may need reduction and simplification

Uncertainty in SISO Systems

Classes of uncertainty

1. **Parametric (real) uncertainty.** Here the structure of the model (including the order) is known, but some of the parameters are uncertain. Sometimes referred to as structured uncertainty.

2. **Dynamic (complex - frequency-dependent) uncertainty.** Here the model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process. Any model of a real system will contain this source of uncertainty. Sometimes referred to as unstructured uncertainty.

1. Parametric uncertainty is quantified by assuming that each uncertain parameter α is bounded within some region $[\alpha_{min}, \alpha_{max}]$. That is,

$$\alpha_p = \bar{\alpha}(1 + r_\alpha \Delta) \tag{5.1}$$

where $\bar{\alpha}$ is the mean value of the parameter, r_α is the relative uncertainty of the parameter ($r_\alpha \triangleq (\alpha_{max} - \alpha_{min})/(\alpha_{max} + \alpha_{min})$), and Δ is a real scalar satisfying $|\Delta| \leq 1$.

2. Dynamic uncertainty is somewhat less precise and thus more difficult to quantify, but it appears that the frequency domain is particularly well suited for this class. This leads to complex uncertainties normalized such that $\|\Delta\|_\infty \leq 1$. Note that parametric uncertainties could be modeled as dynamic uncertainties by restricting Δ to be real (see Section 7.3 in the book).

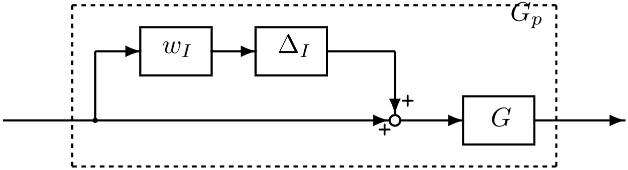

Figure 1: Plant with multiplicative uncertainty

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Multiplicative uncertainty of the form

$$\Pi_I : \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s));$$

where

$$\underbrace{|\Delta_I(j\omega)| \leq 1 \ \forall\omega}_{\|\Delta_I\|_\infty \leq 1} \tag{5.2}$$

Here $\Delta_I(s)$ is *any* stable transfer function which at each frequency is less than or equal to one in magnitude. Some allowable $\Delta_I(s)$'s:

$$\frac{s-z}{s+z}, \quad \frac{1}{\tau s + 1}, \quad \frac{1}{(5s+1)^3}, \quad \frac{0.1}{s^2 + 0.1s + 1}$$

Inverse multiplicative uncertainty

$$\Pi_{iI} : \quad G_p(s) = G(s)(1 + w_{iI}(s)\Delta_{iI}(s))^{-1}; \quad |\Delta_{iI}(j\omega)| \leq 1 \ \forall\omega$$

Representing uncertainty in the frequency domain [7.4]

Parametric uncertainty is also often represented by complex perturbations in order to simplify analysis and control synthesis.

Uncertainty regions [7.4.1]:

Example

$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \le k, \theta, \tau \le 3 \tag{5.3}$$

Uncertainty in SISO Systems

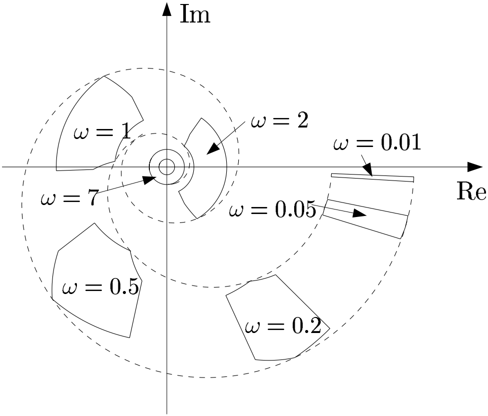

Figure 2: Uncertainty regions of the Nyquist plot at given frequencies. Data from (5.3).

Representing uncertainty region by complex perturbations [7.4.2]:

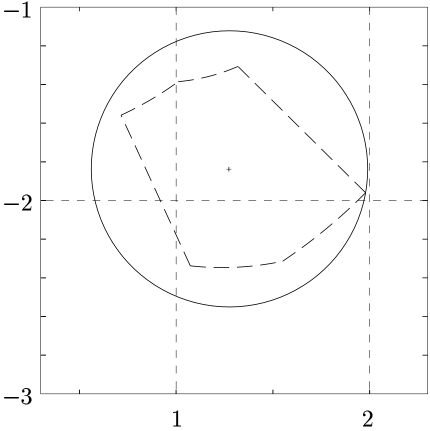

Figure 3: Disc approximation (solid line) of the original uncertainty region (dashed line). Plot corresponds to $\omega = 0.2$ in Figure 2. This is a conservative approach!

Approximation by complex perturbations [7.4.2]

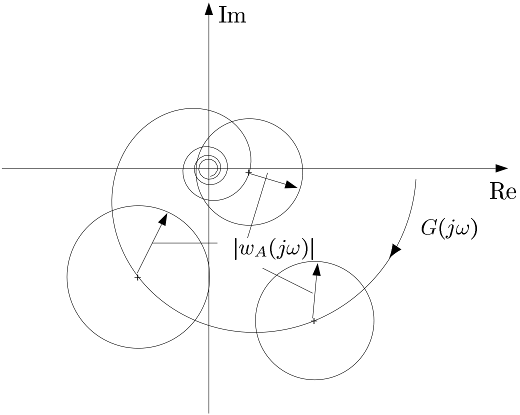

Figure 4: Disc-shaped uncertainty regions generated by complex additive uncertainty, $G_p = G + w_A\Delta$.

Uncertainty in SISO Systems

We use disc-shaped regions to represent uncertainty regions (Figures 3 and 4) generated by *additive uncertainty*

$$\Pi_A : \quad G_p(s) = G(s) + w_A(s)\Delta_A(s); \quad |\Delta_A(j\omega)| \leq 1 \ \forall\omega \tag{5.4}$$

where $\Delta_A(s)$ is *any* stable transfer function which at each frequency is no larger than one in magnitude.

Alternative: *multiplicative uncertainty* description as in (5.2),

$$\Pi_I : \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s)); \quad |\Delta_I(j\omega)| \le 1, \forall\omega \tag{5.5}$$

(5.4) and (5.5) are equivalent if at each frequency

$$|w_I(j\omega)| = |w_A(j\omega)|/|G(j\omega)| \tag{5.6}$$

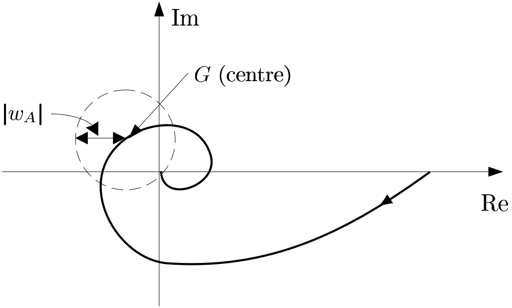

Figure 5: The set of possible plants includes the origin at frequencies where $|w_A(j\omega)| \ge |G(j\omega)|$, or equivalently $|w_I(j\omega)| \ge 1$.

Obtaining the weight for complex uncertainty [7.4.3]

1. Select a nominal model $G(s)$.

2. *Additive uncertainty.* At each frequency find the smallest radius $l_A(\omega)$ which includes all the possible plants Π:

$$|w_A(jw)| \geq l_A(\omega) = \max_{G_P \in \Pi} |G_p(j\omega) - G(j\omega)| \tag{5.7}$$

3. *Multiplicative (relative) uncertainty.* (preferred uncertainty form)

$$|w_I(jw)| \geq l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right| \tag{5.8}$$

Example

Multiplicative weight for parametric uncertainty. *Consider again the set of plants with parametric uncertainty given in (5.3)*

$$\Pi: \quad G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3 \tag{5.9}$$

We want to represent this set using multiplicative uncertainty with a rational weight $w_I(s)$*. We select a delay-free nominal model*

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5 s + 1} \tag{5.10}$$

Uncertainty in SISO Systems

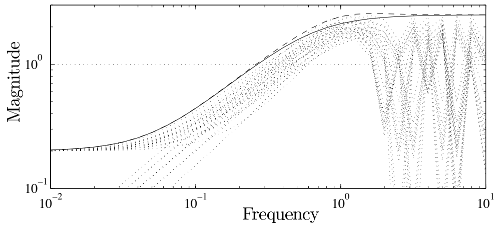

Figure 6: Relative errors for 27 combinations of k, τ and θ with delay-free nominal plant (dotted lines). Solid line: First-order weight $|w_{I1}|$ in (5.11). Dashed line: Third-order weight $|w_I|$ in (5.12).

$$w_{I1}(s) = \frac{Ts + 0.2}{(T/2.5)s + 1}, \quad T = 4 \tag{5.11}$$

$$w_I(s) = \omega_{I1}(s)\ \frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1} \tag{5.12}$$

SISO Robust stability [7.5]

RS with multiplicative uncertainty [7.5.1]:

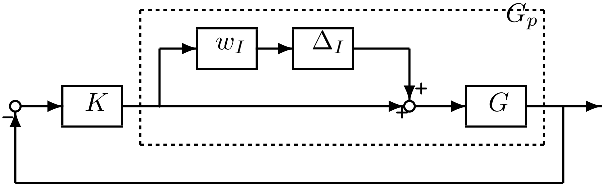

Figure 7: Feedback system with multiplicative uncertainty

1. Graphical derivation of RS-condition.
In Figure 8 $|-1-L| = |1+L|$ is the distance from the point -1 to the centre of the disc representing L_p, $|w_I L|$ is the radius of the disc. Encirclements are avoided if none of the discs cover -1, and we get from Figure 8

$$
\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |w_I L| < |1+L|, \quad \forall\omega &(5.13)\\
&\Leftrightarrow \quad \left|\frac{w_I L}{1+L}\right| < 1, \forall\omega \Leftrightarrow |w_I T| < 1, \forall\omega &(5.14)\\
&\overset{\text{def}}{\Leftrightarrow} \quad \|w_I T\|_\infty < 1 &(5.15)
\end{aligned}
$$

$$\boxed{\text{RS} = |T| < 1/|w_I|, \quad \forall\omega} \qquad (5.16)$$

Uncertainty in SISO Systems

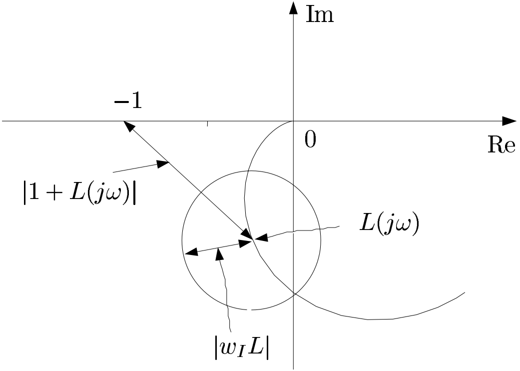

Figure 8: Nyquist plot of L_p for robust stability

2. Algebraic derivation of RS-condition. L_p is assumed stable, and the nominal closed loop is stable. Thus, the nominal loop transfer L does not encircle -1. Encirclements are avoided if none of the plants in L_p go through -1:

$$
\begin{aligned}
\mathrm{RS} \quad &\Leftrightarrow \quad |1+L_p| \neq 0, \quad \forall L_p, \forall \omega && (5.17)\\
&\Leftrightarrow \quad |1+L_p| > 0, \quad \forall L_p, \forall \omega && (5.18)\\
&\Leftrightarrow \quad |1+L+w_I L \Delta_I| > 0, \quad \forall |\Delta_I| \leq 1, \forall \omega && (5.19)
\end{aligned}
$$

At each frequency, worst case scenario: $|\Delta_I| = 1$ and phase s.t. $(1+L)$ and $(w_I L \Delta_I)$ have opposite signs. Thus

$$\boxed{\mathrm{RS} \Leftrightarrow |1+L| - |w_I L| > 0, \quad \forall \omega \quad \Leftrightarrow |w_I T| < 1 \quad \forall \omega} \qquad (5.20)$$

Uncertainty in SISO Systems

Example

Consider the following nominal plant and PI-controller

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)} \quad K(s) = K_c \frac{12.7s+1}{12.7s}$$

$K_c = K_{c1} = 1.13$ *(Ziegler-Nichols).*

One "extreme" uncertain plant is $G'(s) = 4(-3s+1)/(4s+1)^2$*. For this plant the relative error* $|(G'-G)/G|$ *is* 0.33 *at low frequencies; it is* 1 *at about* 0.1 *rad/s, and it is* 5.25 *at high frequencies* $\Rightarrow$ *uncertainty weight*

$$w_I(s) = \frac{10s+0.33}{(10/5.25)s+1}$$

which closely matches this relative error.

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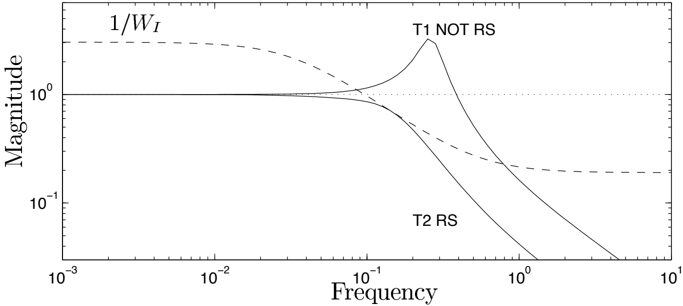

Figure 9: Checking robust stability with multiplicative uncertainty

By trial and error we find that reducing the gain to $K_{c2} = 0.31$ just achieves RS as seen from T_2 in Fig. 9.

Remark:

The procedure is *conservative*. For K_{c2} the system with the "extreme" plant is not at the limit of instability; can increase gain to $k_{c2} = 0.58$ before we get instability.

Uncertainty in SISO Systems

3. $M\Delta$**-structure derivation of RS-condition**.

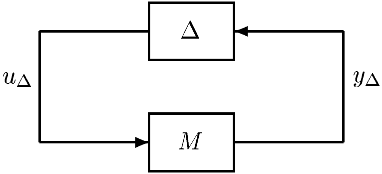

Figure 10: $M\Delta$-structure

The stability of the system in Figure 7 is equivalent to stability of the system in Figure 10, where $\Delta = \Delta_I$ and

$$M = w_I K(1 + GK)^{-1} G = w_I T \tag{5.21}$$

Uncertainty in SISO Systems

The Nyquist stability condition then determines RS if and only if the "loop transfer function" $M\Delta$ does not encircle -1 for all Δ. Thus,

$$\text{RS} \quad \Leftrightarrow \quad |1 + M\Delta| > 0, \quad \forall\omega, \forall|\Delta| \leq 1 \tag{5.22}$$

At each frequency, worst case scenario: $|\Delta| = 1$ and phase s.t. M and 1 have opposite signs. Thus

$$\begin{aligned} \text{RS} \quad &\Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad \forall\omega & (5.23) \\ &\Leftrightarrow \quad |M(j\omega)| < 1, \quad \forall\omega & (5.24) \end{aligned}$$

Uncertainty in SISO Systems

RS with inverse multiplicative uncertainty [7.5.3]:

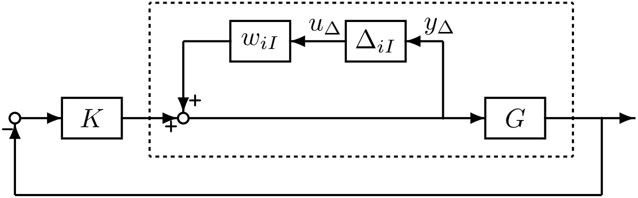

Figure 11: Feedback system with inverse multiplicative uncertainty

$$\boxed{\mathrm{RS} \quad \Leftrightarrow \quad |S| < 1/|w_{iI}|, \quad \forall\omega} \tag{5.25}$$

Uncertainty in SISO Systems

Algebraic derivation of RS-condition. L_p is assumed stable, and the nominal closed loop is stable. Thus, the nominal loop transfer L does not encircle -1. Encirclements are avoided if none of the plants in L_p go through -1:

$$\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |1 + L_p| \neq 0, \quad \forall L_p, \forall \omega && (5.26)\\
&\Leftrightarrow \quad |1 + L_p| > 0, \quad \forall L_p, \forall \omega && (5.27)\\
&\Leftrightarrow \quad |1 + L(1 + w_{iI}\Delta_{iI})^{-1}| > 0, \quad \forall |\Delta_{iI}| \leq 1, \forall \omega && (5.28)\\
&\Leftrightarrow \quad |1 + w_{iI}\Delta_{iI} + L| > 0, \quad \forall |\Delta_{iI}| \leq 1, \forall \omega && (5.29)
\end{aligned}$$

At each frequency, worst case scenario: $|\Delta_{iI}| = 1$ and phase s.t. $(1 + L)$ and $(w_{iI}\Delta_{iI})$ have opposite signs. Thus

$$\boxed{\text{RS} \Leftrightarrow |1 + L| - |w_{iI}| > 0, \quad \forall \omega \quad \Leftrightarrow |w_{iI} S| < 1 \quad \forall \omega} \qquad (5.30)$$

SISO Robust performance [7.6]

Nominal performance in the Nyquist plot [7.6.1]:

$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1+L| \quad \forall\omega \tag{5.31}$$

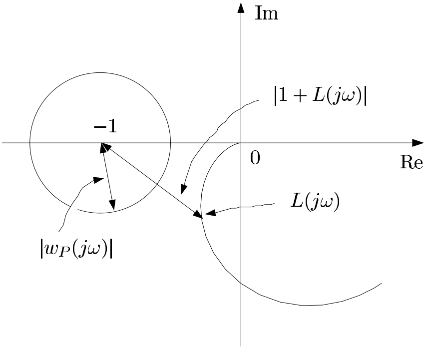

Figure 12: Nyquist plot illustration of nominal performance condition $|w_P| < |1+L|$.

Uncertainty in SISO Systems

Robust performance [7.6.2]:

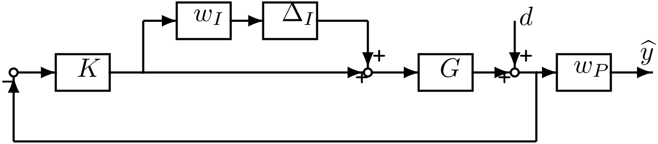

Figure 13: Diagram for robust performance with multiplicative uncertainty

For robust performance we require the performance condition (5.31) to be satisfied for *all* possible plants, that is, including the worst-case uncertainty.

$$\text{RP} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad |w_P S_p| < 1 \quad \forall S_p, \forall \omega \tag{5.32}$$

$$= \quad |w_P| < |1 + L_p| \quad \forall L_p, \forall \omega \tag{5.33}$$

This corresponds to requiring $|\widehat{y}/d| < 1 \; \forall \Delta_I$ in Figure 13, where we consider multiplicative uncertainty, and the set of possible loop transfer functions is

$$L_p = G_p K = L(1 + w_I \Delta_I) = L + w_I L \Delta_I \tag{5.34}$$

1. Graphical derivation of RP-condition. (Figure 14)

$$\begin{aligned} \mathrm{RP} &= |w_P| + |w_I L| < |1+L|, \quad \forall\omega & (5.35)\\ &= |w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1, \forall\omega & (5.36) \end{aligned}$$

$$\boxed{\mathrm{RP} \;=\; \max_\omega \left(|w_P S| + |w_I T|\right) < 1} \quad (5.37)$$

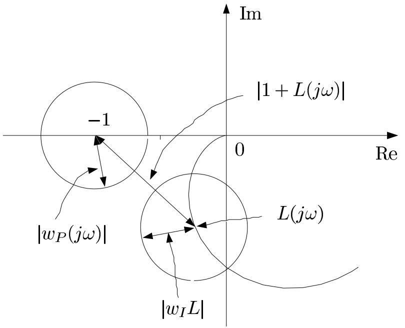

Figure 14: Nyquist plot illustration of robust performance condition $|w_P| < |1 + L_p|$

2. Algebraic derivation of RP-condition. From condition (5.32)

$$\mathrm{RP} \quad \overset{\text{def}}{\Leftrightarrow} \quad \max_{S_p} |w_P S_p| < 1 \quad \forall S_p, \forall \omega \tag{5.38}$$

The perturbed sensitivity is written as $S_p = (1 + L_p)^{-1} = (1 + L + w_I L \Delta_I)^{-1}$. At each frequency, worst case (maximum) scenario: $|\Delta_I| = 1$ and phase s.t. $(1 + L)$ and $(w_I L \Delta_I)$ have opposite signs. Thus,

$$\begin{aligned}
\mathrm{RP} \quad &\Leftrightarrow \quad \max_{S_p} |w_P S_p| < 1, \quad \forall S_p, \forall \omega && (5.39)\\
&\Leftrightarrow \quad \frac{|w_p|}{|1 + L| - |w_I L|} < 1, \quad \forall \omega && (5.40)\\
&\Leftrightarrow \quad \frac{|w_p S|}{1 - |w_I T|} < 1, \quad \forall \omega && (5.41)\\
&\Leftrightarrow \quad |w_p S| + |w_I T| < 1, \quad \forall \omega && (5.42)
\end{aligned}$$

$$\boxed{\mathrm{RP} \Leftrightarrow |w_p S| + |w_I T| < 1 \quad \forall \omega} \tag{5.43}$$

Uncertainty in SISO Systems

The relationship between NP, RS and RP [7.6.3]

$$\text{NP} = |w_P S| < 1, \forall\omega \tag{5.44}$$

$$\text{RS} = |w_I T| < 1, \forall\omega \tag{5.45}$$

$$\text{RP} = |w_P S| + |w_I T| < 1, \forall\omega \tag{5.46}$$

- A prerequisite for RP is that we satisfy NP and RS. This applies in general, both for SISO and MIMO systems and for any uncertainty.

Uncertainty in SISO Systems

- For SISO systems, if we satisfy both RS and NP, then we have at each frequency
$$|w_P S| + |w_I T| \leq 2\max\{|w_P S|, |w_I T|\} < 2 \tag{5.47}$$
Therefore, within a factor of at most 2, we will automatically get RP when NP and RS are satisfied.
- $$|w_P S| + |w_I T| \geq \min\{|w_P|, |w_I|\} \tag{5.48}$$
We *cannot have both* $|w_P| > 1$ *(i.e. good performance) and* $|w_I| > 1$ *(i.e. more than* 100% *uncertainty) at the same frequency.*