Multivariable Robust Control

Lecture 4 (Meeting 8)

Chapter 3: Introduction to Multivariable Control

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Introduction [3.1]

We consider a MIMO plant with m inputs and l outputs. Thus, the basic transfer function model is

$$y(s) = G(s)u(s)$$

where y is an $l \times 1$ vector, u is an $m \times 1$ vector, and $G(s)$ is an $l \times m$ transfer function matrix.

- MIMO systems show *interaction* between inputs and outputs. This means that one input may affect all the outputs.
- The main difference between SISO and MIMO systems is the presence of *directions* in the MIMO systems.
- Most ideas and techniques valid for SISO systems can be extended to MIMO systems.
- The singular value decomposition (SVD) provides a useful way of quantifying *multivariable directionality*.
  - SISO: absolute value (magnitude) $\rightarrow$ MIMO: maximum singular value
  - Exception: Bode's stability condition (no generalization in terms of singular values)

Transfer functions for MIMO systems [3.2]

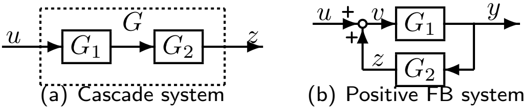


Figure 1: Block diagrams for the cascade rule and the feedback rule

1. **Cascade rule.** (Figure 1(a)) $G = G_2 G_1$
2. **Feedback rule.** (Figure 1(b)) $v = (I - L)^{-1} u$ where $L = G_2 G_1$
3. **Push-through rule.**

$$G_1 (I - G_2 G_1)^{-1} \quad = \quad (I - G_1 G_2)^{-1} G_1$$

NOTE: Verified by premultiplying $(I - G_1 G_2)$ and postmultiplying by $(I - G_2 G_1)$.

MIMO Rule: *Start from the output, move backwards. If you exit from a feedback loop then include a term $(I - L)^{-1}$ where L is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).*

Example

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \tag{4.1}$$

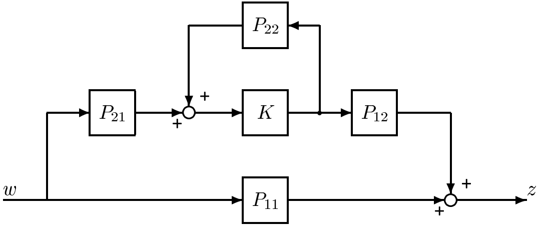


Figure 2: Block diagram corresponding to (4.1)

Negative feedback control systems

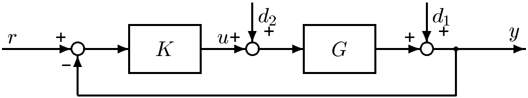


Figure 3: Conventional negative feedback control system

- L is the loop transfer function when breaking the loop at *output* of the plant.

$$L = GK \tag{4.2}$$

Accordingly

$$S \triangleq (I + L)^{-1} \quad \textit{output sensitivity} \tag{4.3}$$

$$T \triangleq I - S = (I + L)^{-1} L = L(I + L)^{-1} \quad \textit{output complementary sensitivity} \tag{4.4}$$

$L_O \equiv L$, $S_O \equiv S$ and $T_O \equiv T$.

- L_I is the loop transfer function at the *input* to the plant

$$L_I = KG \tag{4.5}$$

Input sensitivity:

$$S_I \triangleq (I + L_I)^{-1}$$

Input complementary sensitivity:

$$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$

- Some relationships:

$$(I+L)^{-1} + (I+L)^{-1}L = S + T = I \tag{4.6}$$

$$G(I+KG)^{-1} = (I+GK)^{-1}G \tag{4.7}$$

$$GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK \tag{4.8}$$

$$T = L(I+L)^{-1} = (I+L^{-1})^{-1} = (I+L)^{-1}L \tag{4.9}$$

Rule to remember: "G comes first and then G and K alternate in sequence".

Multivariable frequency response [3.3]

Obtaining the frequency response from $G(s)$ [3.3.1]:

$$\begin{aligned} G(s) &= \text{transfer (function) matrix} \\ G(j\omega) &= \text{complex matrix representing response} \\ &\quad \text{to sinusoidal signal of frequency } \omega \end{aligned}$$

Note: $d \in R^m$ and $y \in R^l$

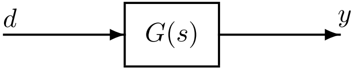


Figure 4: System $G(s)$ with input d and output y

$$y(s) = G(s)d(s) \tag{4.10}$$

Sinusoidal input to channel j

$$d_j(t) = d_{j0}\sin(\omega t + \alpha_j) \tag{4.11}$$

starting at $t = -\infty$. Output in channel i is a sinusoid with the same frequency

$$y_i(t) = y_{i0}\sin(\omega t + \beta_i) \tag{4.12}$$

Amplification (gain):

$$\frac{y_{io}}{d_{jo}} = |g_{ij}(j\omega)| \tag{4.13}$$

Phase shift:

$$\beta_i - \alpha_j = \angle g_{ij}(j\omega) \tag{4.14}$$

$g_{ij}(j\omega)$ represents the sinusoidal response from input j to output i.

Example: 2×2 multivariable system, sinusoidal signals of the same frequency ω to the two input channels:

$$d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10}\sin(\omega t + \alpha_1) \\ d_{20}\sin(\omega t + \alpha_2) \end{bmatrix} \tag{4.15}$$

The output signal

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10}\sin(\omega t + \beta_1) \\ y_{20}\sin(\omega t + \beta_2) \end{bmatrix} \tag{4.16}$$

can be computed by multiplying complex matrix $G(j\omega)$ by complex vector $d(\omega)$:

$$\begin{aligned} y(\omega) &= G(j\omega)d(\omega) \\ y(\omega) &= \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \; d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix} \end{aligned} \tag{4.17}$$

Directions in multivariable systems [3.3.2]:

SISO system ($y = Gd$): gain

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on ω, but is independent of $|d(\omega)|$.

MIMO system: input and output are vectors.
$\Rightarrow$ need to "sum up" magnitudes of elements in each vector by use of some norm

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \tag{4.18}$$

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots} \tag{4.19}$$

The *gain* of the system $G(s)$ is

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}} \tag{4.20}$$

The gain depends on ω, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the ***direction*** of the input d.

Example: Consider the five inputs (all $\|d\|_2 = 1$)

$$\begin{aligned} d_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \\ d_4 &= \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix},\ d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix} \end{aligned}$$

For the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{4.21}$$

The five inputs d_j lead to the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},\ y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix},\ y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

with the 2-norms (i.e. the gains for the five inputs)

$$\|y_1\|_2 = 5.83,\ \|y_2\|_2 = 4.47,\ \|y_3\|_2 = 7.30, \|y_4\|_2 = 1.00,\ \|y_5\|_2 = 0.28$$

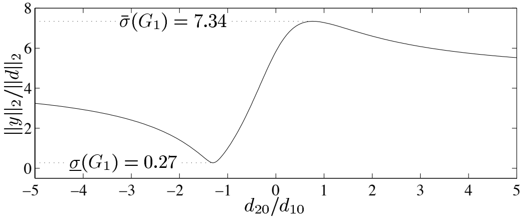


Figure 5: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (4.21)

The maximum value of the gain in (4.20) as the direction of the input is varied, is the maximum singular value of G,

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \tag{4.22}$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \tag{4.23}$$

NOTE: The first identities hold because the gain is independent of the input magnitude.

Eigenvalues are a poor measure of gain [3.3.3]:

Example:

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \quad G \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{4.24}$$

Both eigenvalues are equal to zero, but gain is equal to 100.

Problem: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).

For generalizations of $|G|$ when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties: *triangle inequality*

$$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\| \tag{4.25}$$

and the multiplicative property

$$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\| \tag{4.26}$$

$\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm

Singular value decomposition [3.3.4]

Any matrix G may be decomposed into its singular value decomposition (H denotes Hermitian transpose),

$$G = U\Sigma V^H \tag{4.27}$$

Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal;

U is an $l \times l$ unitary matrix of output singular vectors, u_i,

V is an $m \times m$ unitary matrix of input singular vectors, v_i,

$$\sigma_i(G) = \sqrt{\lambda_i(G^H G)} = \sqrt{\lambda_i(GG^H)} \tag{4.28}$$

$$(GG^H)U = U\Sigma\Sigma^H, \qquad (G^H G)V = V\Sigma^H\Sigma \tag{4.29}$$

Example: SVD of a real 2×2 matrix can always be written as

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T} \qquad (4.30)$$

U and V involve rotations and their columns are orthonormal.

Input and output directions. The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

$$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1 \tag{4.31}$$

$$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j \tag{4.32}$$

The column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*.

$$G = U \Sigma V^H \Rightarrow GV = U\Sigma \quad (V^H V = I) \Rightarrow G v_i = \sigma_i u_i \tag{4.33}$$

If we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ σ_i gives the gain of the matrix G in this direction.

$$\sigma_i(G) = \|G v_i\|_2 = \frac{\|G v_i\|_2}{\|v_i\|_2} \tag{4.34}$$

Maximum and minimum singular values. The largest gain for *any* input direction is

$$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} \tag{4.35}$$

The smallest gain for any input direction is

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} \tag{4.36}$$

where $k = \min\{l, m\}$. For any vector d we have

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \tag{4.37}$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \tag{4.38}$$

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the "strongest", "high-gain" or "most important" directions.

Example:

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{4.39}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$, the smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. Since in (4.39) both inputs affect both outputs, we say that the system is *interactive*.

The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*: $\bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example: Shopping cart. Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards (maximum singular value), sideways (medium singular value) and upwards (minimum singular value). For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned.

Example: **Distillation process.** Steady-state model of a distillation column

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{4.40}$$

Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{4.41}$$

The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$.

Singular values for performance [3.3.5]:

Maximum singular value is very useful in terms of frequency-domain performance and robustness.

Performance measure for SISO systems:

$$|e(\omega)|/|r(\omega)| = |S(j\omega)|$$

Generalization for MIMO systems $\|e(\omega)\|_2/\|r(\omega)\|_2$

$$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega)) \tag{4.42}$$

For *performance* we want the gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ small for any direction of $r(\omega)$

$$\begin{aligned} \bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|,\ \forall\omega \quad &\Leftrightarrow \quad \bar{\sigma}(w_P S) < 1, \forall\omega \\ &\Leftrightarrow \quad \|w_P S\|_\infty < 1 \end{aligned} \tag{4.43}$$

where the $\mathcal{H}_\infty$ norm is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega)) \tag{4.44}$$

Typical singular values of $S(j\omega)$ in Figure 6.

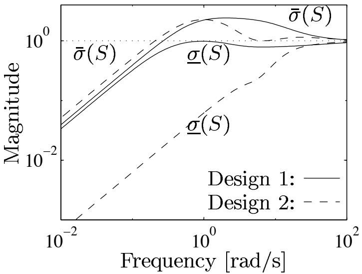


Figure 6: Singular values of S for a 2×2 plant with RHP-zero

- *Bandwidth*, ω_B: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below. Since $S = (I+L)^{-1}$, the singular values inequality

$$\underline{\sigma}(A) - 1 \leq \frac{1}{\bar{\sigma}(I+A)^{-1}} \leq \underline{\sigma}(A) + 1$$

yields

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \tag{4.45}$$

- low ω : $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$
- high ω: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$

Introduction to MIMO robustness [3.7]

Motivating robustness example no. 1: Spinning Satellite [3.7.1]:

Angular velocity control of a satellite spinning about one of its principal axes:

$$G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}; \quad a = 10 \tag{4.46}$$

A minimal, state-space realization, $G = C(sI - A)^{-1}B + D$, is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right] \tag{4.47}$$

Poles at $s = \pm ja$ For stabilization:

$$K = I$$

$$T(s) = GK(I+GK)^{-1} = \frac{1}{s+1}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \tag{4.48}$$

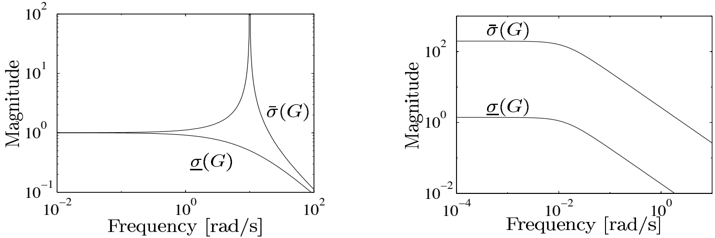
Note that $T_{11}(s) = \frac{1}{s+1} \stackrel{\Delta}{=} \frac{L_1(s)}{1+L_1(s)}$. Therefore $L_1(s) = \frac{1}{s}$.

Nominal stability (NS). Two closed loop poles at $s = -1$ and

$$A_{cl} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Stable!

Nominal performance (NP). Figure 7(a)



(a) Spinning satellite in (4.46)   (b) Distillation process in (4.51)

Figure 7: Typical plots of singular values

- $\underline{\sigma}(L) \leq 1 \quad \forall \omega$ poor performance in low gain direction
- T_{12}, T_{21} large $\Rightarrow$ strong interaction

Robust stability (RS).
Check stability: one loop at a time.

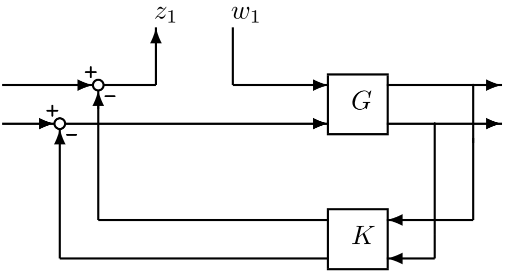


Figure 8: Checking stability margins "one-loop-at-a-time"

$$\frac{z_1}{w_1} \triangleq L_1(s) = \frac{1}{s} \Rightarrow GM = \infty, PM = 90^\circ \tag{4.49}$$

- Good Robustness? NO
- Consider perturbation in each feedback channel

$$u_1' = (1+\epsilon_1)u_1, \quad u_2' = (1+\epsilon_2)u_2 \tag{4.50}$$

$$B' = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$$

Closed-loop state matrix:

$$A'_{cl} = A - B'KC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

Characteristic polynomial:

$$\det(sI - A'_{cl}) = s^2 + \underbrace{(2 + \epsilon_1 + \epsilon_2)}_{a_1} s + \\ + \underbrace{1 + \epsilon_1 + \epsilon_2 + (a^2 + 1)\epsilon_1\epsilon_2}_{a_0}$$

Coefficients must be positive for stability.

- Let us consider uncertainty in only one channel at a time. Stability for $(-1 < \epsilon_1 < \infty, \epsilon_2 = 0)$ and $(\epsilon_1 = 0, -1 < \epsilon_2 < \infty)$ (GM=∞).
- But only *small simultaneous changes* in the two channels: for example, let $\epsilon_1 = -\epsilon_2$, then the system is unstable $(a_0 < 0)$ for

$$|\epsilon_1| > \frac{1}{\sqrt{a^2 + 1}} \approx 0.1$$

Summary. Checking single-loop margins is inadequate for MIMO problems.

Motivating robustness example no. 2: Distillation Process [3.7.2]:

Idealized dynamic model of a distillation column,

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{4.51}$$

The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$.

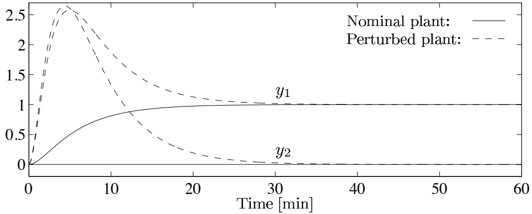


Figure 9: Response with decoupling controller to filtered reference input $r_1 = 1/(5s+1)$. The perturbed plant has 20% gain uncertainty as given by (4.54).

Inverse-based controller or equivalently steady-state decoupler with a PI controller $(k_1 = 0.7)$

$$K_{\text{inv}}(s) = \frac{k_1}{s} G^{-1}(s) = \frac{k_1(1+75s)}{s} \begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \tag{4.52}$$

Nominal performance (NP).

$$GK_{\text{inv}} = K_{\text{inv}} G = \frac{0.7}{s} I$$

first order response with time constant 1.43 (Fig. 9).
Nominal performance (NP) achieved with decoupling controller.

Robust stability (RS).

$$S = S_I = \frac{s}{s+0.7} I; \quad T = T_I = \frac{1}{1.43s+1} I \tag{4.53}$$

In each channel: GM$=\infty$, PM$=90^\circ$.

Input gain uncertainty (4.50) with $\epsilon_1 = 0.2$ and $\epsilon_2 = -0.2$:

$$u_1' = 1.2u_1, \quad u_2' = 0.8u_2 \tag{4.54}$$

$$\begin{aligned} L_I'(s) &= K_{\text{inv}} G' = K_{\text{inv}} G \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} = \\ & \frac{0.7}{s} \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \end{aligned} \tag{4.55}$$

Perturbed closed-loop poles are

$$s_1 = -0.7(1+\epsilon_1), \quad s_2 = -0.7(1+\epsilon_2) \tag{4.56}$$

Closed-loop stability as long as the input gains $1+\epsilon_1$ and $1+\epsilon_2$ remain positive $\Rightarrow$ *Robust stability (RS) achieved with respect to input gain errors for the decoupling controller*.

Robust performance (RP).

Performance with input gain errors is poor (Fig. 9)

- SISO: NP+RS $\Rightarrow$ RP
- MIMO: NP+RS $\not\Rightarrow$ RP

RP is not achieved by the decoupling controller.

Robustness conclusions [3.7.3]: Multivariable plants can display a sensitivity to uncertainty (in this case input uncertainty) which is fundamentally different from what is possible in SISO systems.

General control problem formulation [3.8]

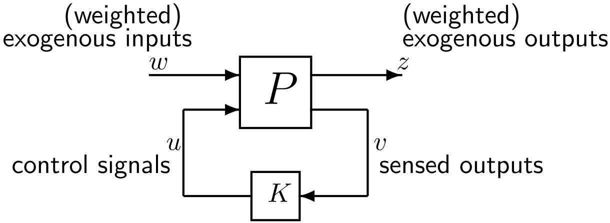


Figure 10: General control configuration for the case with no model uncertainty

The overall control objective is to minimize some norm of the transfer function from w to z, for example, the $\mathcal{H}_\infty$ norm. The controller design problem is then:

Find a controller K which based on the information in v, generates a control signal u which counteracts the influence of w on z, thereby minimizing the closed-loop norm from w to z.

Obtaining the generalized plant P [3.8.1]:

The routines in MATLAB for synthesizing $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers assume that the problem is in the general form of Figure 10

Example: One degree-of-freedom feedback control configuration.

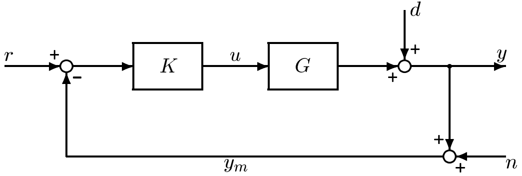


Figure 11: One degree-of-freedom control configuration

Equivalent representation of Figure 11 where the error signal to be minimized is $z = y - r$ and the input to the controller is $v = r - y_m$

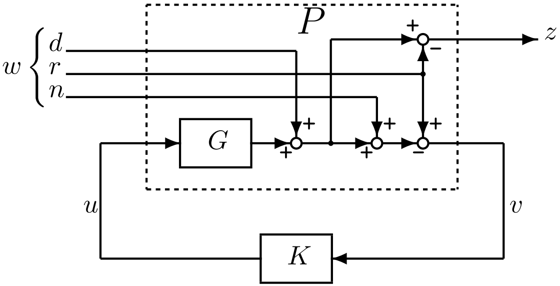


Figure 12: General control configuration equivalent to Figure 11

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; z = e = y - r; v = r - y_m = r - y - n \tag{4.57}$$

$$\begin{aligned} z &= y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu \\ v &= r - y_m = r - Gu - d - n = \\ &= -Iw_1 + Iw_2 - Iw_3 - Gu \end{aligned}$$

and P which represents the transfer function matrix from $\begin{bmatrix} w & u \end{bmatrix}^T$ to $\begin{bmatrix} z & v \end{bmatrix}^T$ is

$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \tag{4.58}$$

Note that P does *not* contain the controller. Alternatively, P can be obtained from Figure 12.

Remark. In MATLAB we may obtain P via `simulink`, or we may use the `sysic` program in the μ-toolbox. The code in Table 1 generates the generalized plant P in (4.58) for Figure 11.

Table 1: **MATLAB program to generate P**

```
% Uses the Mu-toolbox
systemnames = 'G';                    % G is the SISO plant.
inputvar = '[d(1);r(1);n(1);u(1)]'; % Consists of vectors w and u.
input_to_G = '[u]';
outputvar = '[G+d-r; r-G-d-n]';      % Consists of vectors z and v.
sysoutname = 'P';
sysic;
```

Including weights in P [3.8.2]:

To get a meaningful controller synthesis problem, for example, in terms of the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms, we generally have to include weights W_z and W_w in the generalized plant P, see Figure 13.

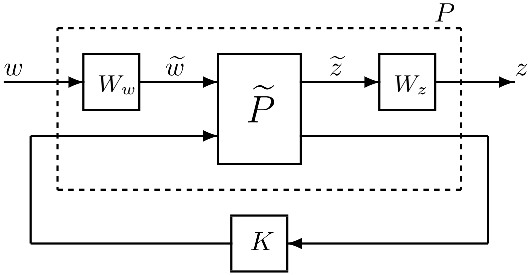


Figure 13: General control configuration for the case with no model uncertainty

That is, we consider the weighted or normalized exogenous inputs w, and the weighted or normalized controlled outputs $z = W_z \widetilde{z}$. The weighting matrices are usually frequency dependent and typically selected such that weighted signals w and z are of magnitude 1, that is, the norm from w to z should be less than 1.

Example: Stacked $S/T/KS$ problem.

Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem

$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix} \tag{4.59}$$

where K is a stabilizing controller. In other words, we have $z = Nw$ and the objective is to minimize the $\mathcal{H}_\infty$ norm from w to z.

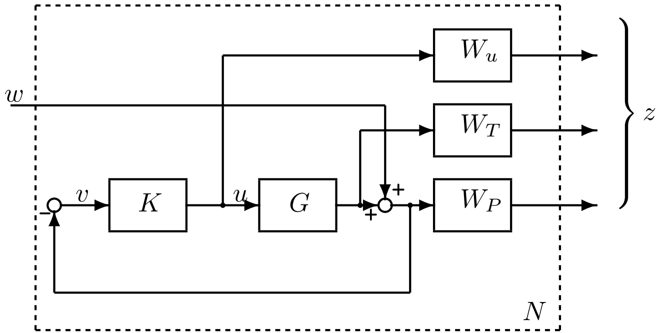


Figure 14: Block diagram corresponding to generalized plant in (4.59)

$$
\begin{aligned}
z_1 &= W_u u \\
z_2 &= W_T G u \\
z_3 &= W_P w + W_P G u \\
v &= -w - Gu
\end{aligned}
$$

so the generalized plant P from $\begin{bmatrix} w & u \end{bmatrix}^T$ to $\begin{bmatrix} z & v \end{bmatrix}^T$ is

$$
P = \left[\begin{array}{c|c} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ \hline -I & -G \end{array}\right] \tag{4.60}
$$

Partitioning the generalized plant P [3.8.3]:

We often partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{4.61}$$

so that

$$z = P_{11}w + P_{12}u \tag{4.62}$$

$$v = P_{21}w + P_{22}u \tag{4.63}$$

In Example "Stacked $S/T/KS$ problem" we get from (4.60)

$$P_{11} = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} \tag{4.64}$$

$$P_{21} = -I, \quad P_{22} = -G \tag{4.65}$$

Note that P_{22} has dimensions compatible with the controller K in Figure 13

Analysis: Closing the loop to get N [3.8.4]:

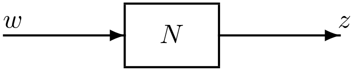


Figure 15: General block diagram for analysis with no uncertainty

For *analysis* of closed-loop performance we may absorb K into the interconnection structure and obtain the system N as shown in Figure 15 where

$$z = Nw \tag{4.66}$$

where N is a function of K.

To find N, first partition the generalized plant P as given in (4.61)-(4.63), combine this with the controller equation

$$u = Kv \tag{4.67}$$

and eliminate u and v from equations (4.62), (4.63) and (4.67) to yield $z = Nw$ where N is given by

$$N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq F_l(P, K) \tag{4.68}$$

Here $F_l(P, K)$ denotes a lower *linear fractional transformation (LFT)* of P with K as the parameter. In words, N is obtained from Figure 10 by using K to close a lower feedback loop around P. Since positive feedback is used in the general configuration in Figure 10 the term $(I - P_{22}K)^{-1}$ has a negative sign.

Example: We want to derive N for the partitioned P in (4.64) and (4.65) using the LFT-formula in (4.68). We get

$$N = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix} + \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} K(I+GK)^{-1}(-I) = \begin{bmatrix} -W_u KS \\ -W_T T \\ W_P S \end{bmatrix}$$

where we have made use of the identities $S = (I+GK)^{-1}$, $T = GKS$ and $I - T = S$.

In the MATLAB μ-Toolbox we can evaluate $N = F_l(P,K)$ using the command `N=starp(P,K)`. Here `starp` denotes the matrix star product which generalizes the use of LFTs.

Further examples [3.8.5]:

Example: Consider the control system in Figure 16, where y_1 is the output we want to control, y_2 is a secondary output (extra measurement), and we also measure the disturbance d. The control configuration includes a two degrees-of-freedom controller, a feedforward controller and a local feedback controller based on the extra measurement y_2.

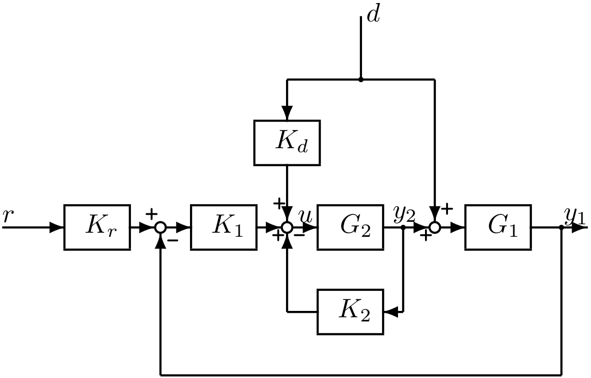


Figure 16: System with feedforward, local feedback and two degrees-of-freedom control

To recast this into our standard configuration of Figure 10 we define

$$w = \begin{bmatrix} d \\ r \end{bmatrix}; \quad z = y_1 - r; \quad v = \begin{bmatrix} r \\ y_1 \\ y_2 \\ d \end{bmatrix} \tag{4.69}$$

$$K = \begin{bmatrix} K_1 K_r & -K_1 & -K_2 & K_d \end{bmatrix} \tag{4.70}$$

We get

$$P = \left[\begin{array}{cc|c} G_1 & -I & G_1 G_2 \\ \hline 0 & I & 0 \\ G_1 & 0 & G_1 G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{array}\right] \tag{4.71}$$

Then partitioning P as in (4.62) and (4.63) yields
$P_{22} = \begin{bmatrix} 0^T & (G_1 G_2)^T & G_2^T & 0^T \end{bmatrix}^T$.

Deriving P from N [3.8.6]:

For cases where N is given and we wish to find a P such that

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

it is usually best to work from a block diagram representation. This was illustrated above for the stacked N in (4.59). Alternatively, the following procedure may be useful:

1. Set $K = 0$ in N to obtain P_{11}.
2. Define $Q = N - P_{11}$ and rewrite Q such that each term has a common factor $R = K(I - P_{22}K)^{-1}$ (this gives P_{22}).
3. Since $Q = P_{12}RP_{21}$, we can now usually obtain P_{12} and P_{21} by inspection.

Example

Weighted sensitivity. *We will use the above procedure to derive P when*
$N = w_P S = w_P(I + GK)^{-1}$,
where w_P is a scalar weight.

1. $P_{11} = N(K = 0) = w_P I$.
2. $Q = N - w_P I = w_P(S - I) = -w_P T = -w_P GK(I + GK)^{-1}$, *and we have* $R = K(I + GK)^{-1}$ *so* $P_{22} = -G$.
3. $Q = -w_P GR$ *so we have* $P_{12} = -w_P G$ *and* $P_{21} = I$, *and we get*

$$P = \begin{bmatrix} w_P I & -w_P G \\ I & -G \end{bmatrix} \tag{4.72}$$

A general control configuration including model uncertainty [3.8.8]:

The general control configuration in Figure 10 may be extended to include model uncertainty. Here the matrix Δ is a *block-diagonal* matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$.

Introduction to Multivariable Control

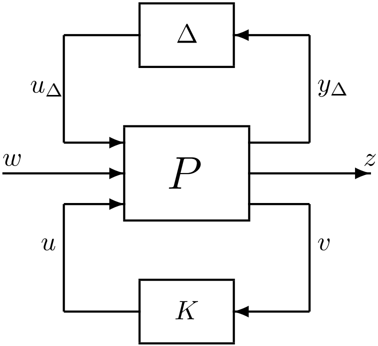


Figure 17: General control configuration for the case with model uncertainty

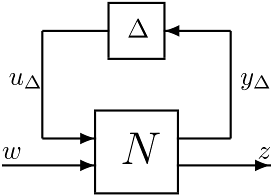


Figure 18: General block diagram for analysis with uncertainty included

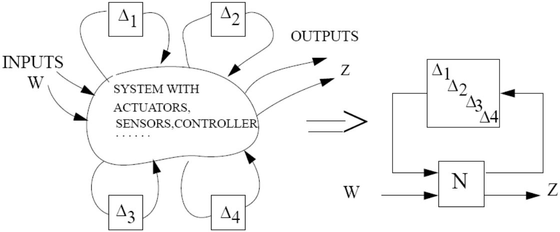


Figure 19: Rearranging a system with multiple perturbations into the $N\Delta$-structure

The block diagram in Figure 17 in terms of P (for synthesis) may be transformed into the block diagram in Figure 18 in terms of N (for analysis) by using K to close a lower loop around P. The same *lower LFT* as found in (4.68) applies, and

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{4.73}$$

To evaluate the perturbed (uncertain) transfer function from external inputs w to external outputs z, we use Δ to close the upper loop around N (see Figure 18), resulting in an *upper LFT*:

$$z = F_u(N, \Delta)w;\ F_u(N, \Delta) \stackrel{\Delta}{=} N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \tag{4.74}$$

Remark 1 Almost any control problem with uncertainty can be represented by Figure 17. First represent each source of uncertainty by a perturbation block, Δ_i, which is normalized such that $\|\Delta_i\| \leq 1$. Then "pull out" each of these blocks from the system so that an input and an output can be associated with each Δ_i as shown in Figure 19(a). Finally, collect these perturbation blocks into a large block-diagonal matrix having perturbation inputs and outputs as shown in Figure 19(b).