Lecture 4

Chapter 3: Introduction to Multivariable Control

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4.1 Transfer functions for MIMO systems [3.2]

Figure 1: Block diagrams for the cascade rule and the feedback rule

1. Cascade rule. (Figure 1(a)) \( G = G_2 G_1 \)

2. Feedback rule. (Figure 1(b)) \( v = (I - L)^{-1} u \) where \( L = G_2 G_1 \)

3. Push-through rule.

\[
G_1 (I - G_2 G_1)^{-1} = (I - G_1 G_2)^{-1} G_1
\]
MIMO Rule: Start from the output, move backwards. If you exit from a feedback loop then include a term \((I - L)^{-1}\) where \(L\) is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).

Example

\[ z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \]

Figure 2: Block diagram corresponding to (4.1)
Figure 3: Conventional negative feedback control system

$L$ is the loop transfer function when breaking the loop at the *output* of the plant.

\[(4.2) \quad L = GK\]
Accordingly

\[ S \triangleq (I + L)^{-1} \]

(4.3) \textit{output sensitivity}

\[ T \triangleq I - S = (I + L)^{-1}L = L(I + L)^{-1} \]

(4.4) \textit{output complementary sensitivity}

\[ L_O \equiv L, \quad S_O \equiv S \quad \text{and} \quad T_O \equiv T. \]
$L_I$ is the loop transfer function at the \textit{input} to the plant

\begin{equation}
L_I = KG
\end{equation}

\textit{Input} sensitivity:

$$S_I \triangleq (I + L_I)^{-1}$$

\textit{Input} complementary sensitivity:

$$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$
Some relationships:

(4.6) \((I + L)^{-1} + (I + L)^{-1}L = S + T = I\)

(4.7) \(G(I + KG)^{-1} = (I + GK)^{-1}G\)

(4.8) \(GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK\)

(4.9) \(T = L(I + L)^{-1} = (I + L^{-1})^{-1} = (I + L)^{-1}L\)

Rule to remember: “\(G\) comes first and then \(G\) and \(K\) alternate in sequence”.
4.2 Multivariable frequency response [3.3]

\[ G(s) = \text{transfer (function) matrix} \]
\[ G(j\omega) = \text{complex matrix representing response to sinusoidal signal of frequency } \omega \]

Note: \( d \in \mathbb{R}^m \) and \( y \in \mathbb{R}^l \)

Figure 4: System \( G(s) \) with input \( d \) and output \( y \)

(4.10) \[ y(s) = G(s)d(s) \]
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Sinusoidal input to channel $j$

\[ d_j(t) = d_{j0} \sin(\omega t + \alpha_j) \]  

(4.11)  

starting at $t = -\infty$. Output in channel $i$ is a sinusoid with the same frequency

\[ y_i(t) = y_{i0} \sin(\omega t + \beta_i) \]  

(4.12)  

Amplification (gain):

\[ \frac{y_{i0}}{d_{j0}} = |g_{ij}(j\omega)| \]  

(4.13)  

Phase shift:

\[ \beta_i - \alpha_j = \angle g_{ij}(j\omega) \]  

(4.14)  

$g_{ij}(j\omega)$ represents the sinusoidal response from input $j$ to output $i$.  

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Example 2 $\times$ 2 multivariable system, sinusoidal signals of the same frequency $\omega$ to the two input channels:

\[
d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10} \sin(\omega t + \alpha_1) \\ d_{20} \sin(\omega t + \alpha_2) \end{bmatrix}
\]

(4.15)

The output signal

\[
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10} \sin(\omega t + \beta_1) \\ y_{20} \sin(\omega t + \beta_2) \end{bmatrix}
\]

(4.16)

can be computed by multiplying the complex matrix $G(j\omega)$ by the complex vector $d(\omega)$:

\[
y(\omega) = G(j\omega)d(\omega)
\]

(4.17)

\[
y(\omega) = \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \quad d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix}
\]
4.2.1 Directions in multivariable systems [3.3.2]

SISO system \((y = Gd)\): gain

\[
\frac{|y(\omega)|}{|d(\omega)|} = \left| \frac{G(j\omega)d(\omega)}{d(\omega)} \right| = |G(j\omega)|
\]

The gain depends on \(\omega\), but is independent of \(|d(\omega)|\).

MIMO system: input and output are vectors.

\(\Rightarrow\) need to “sum up” magnitudes of elements in each vector by use of some norm

\[
(4.18) \quad \|d(\omega)\|_2 = \sqrt{\sum |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots}
\]
The gain of the system $G(s)$ is

\begin{equation}
\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots}
\end{equation}

The gain depends on $\omega$, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the direction of the input $d$. 
Example Consider the five inputs (all $\|d\|_2 = 1$)

\[
\begin{align*}
  d_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
  d_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
  d_3 &= \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \\
  d_4 &= \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, \\
  d_5 &= \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}
\end{align*}
\]

For the $2 \times 2$ system

\[
(4.21) \quad G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}
\]

The five inputs $d_j$ lead to the following output vectors

\[
\begin{align*}
  y_1 &= \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \\
  y_2 &= \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \\
  y_3 &= \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix}, \\
  y_4 &= \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \\
  y_5 &= \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}
\end{align*}
\]

with the 2-norms (i.e. the gains for the five inputs)

\[
\begin{align*}
  \|y_1\|_2 &= 5.83, \\
  \|y_2\|_2 &= 4.47, \\
  \|y_3\|_2 &= 7.30, \\
  \|y_4\|_2 &= 1.00, \\
  \|y_5\|_2 &= 0.28
\end{align*}
\]
Figure 5: Gain $\frac{\|y\|_2}{\|d\|_2}$ as a function of $d_{20}/d_{10}$ for $G_1$ in (4.21)
The maximum value of the gain in (4.20) as the direction of the input is varied, is the maximum singular value of $G$,

$$
\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2 = 1} \|Gd\|_2 = \bar{\sigma}(G) \tag{4.22}
$$

whereas the minimum gain is the minimum singular value of $G$,

$$
\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2 = 1} \|Gd\|_2 = \sigma(G) \tag{4.23}
$$
4.2.2 Eigenvalues are a poor measure of gain [3.3.3]

Example

\[ G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \quad G \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \]

Both eigenvalues are equal to zero, but gain is equal to 100.

**Problem**: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).
For generalizations of $|G|$ when $G$ is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties: *triangle inequality*

\[(4.25) \quad \|G_1 + G_2\| \leq \|G_1\| + \|G_2\|\]

and the multiplicative property

\[(4.26) \quad \|G_1G_2\| \leq \|G_1\| \cdot \|G_2\|\]

$\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm
4.2.3 Singular value decomposition [3.3.4]

Any matrix $G$ may be decomposed into its singular value decomposition,

\begin{equation}
G = U \Sigma V^H
\end{equation}

$\Sigma$ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, $\sigma_i$, arranged in descending order along its main diagonal;

$U$ is an $l \times l$ unitary matrix of output singular vectors, $u_i$,

$V$ is an $m \times m$ unitary matrix of input singular vectors, $v_i$,

\begin{equation}
\sigma_i(G) = \sqrt{\lambda_i(G^H G)} = \sqrt{\lambda_i(GG^H)}
\end{equation}

\begin{equation}
(GG^H)U = U\Sigma\Sigma^H, \quad (G^H G)V = V\Sigma^H\Sigma
\end{equation}
Example  SVD of a real $2 \times 2$ matrix can always be written as

$$(4.30) = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & \pm \sin \theta_2 \\ -\sin \theta_2 & \pm \cos \theta_2 \end{bmatrix}^T$$

$U$ and $V$ involve rotations and their columns are orthonormal.
Input and output directions. The column vectors of $U$, denoted $u_i$, represent the output directions of the plant. They are orthogonal and of unit length (orthonormal), that is

\begin{equation}
\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1
\end{equation}

\begin{equation}
\begin{aligned}
    u_i^H u_i &= 1, \\
    u_i^H u_j &= 0, \quad i \neq j
\end{aligned}
\end{equation}

The column vectors of $V$, denoted $v_i$, are orthogonal and of unit length, and represent the input directions.

\begin{equation}
G = U\Sigma V^H \Rightarrow GV = U\Sigma \quad (V^HV = I) \Rightarrow Gv_i = \sigma_i u_i
\end{equation}

If we consider an input in the direction $v_i$, then the output is in the direction $u_i$. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1 \sigma_i$ gives the gain of the matrix $G$ in this direction.

\begin{equation}
\sigma_i(G) = \|Gv_i\|_2 = \frac{\|Gv_i\|_2}{\|v_i\|_2}
\end{equation}
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Maximum and minimum singular values. The largest gain for any input direction is

\[
\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2}
\]

(4.35)

The smallest gain for any input direction is

\[
\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2}
\]

(4.36)

where \( k = \min\{l, m\} \). For any vector \( d \) we have

\[
\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G)
\]

(4.37)
Define $u_1 = \bar{u}$, $v_1 = \bar{v}$, $u_k = u$ and $v_k = v$. Then

(4.38) \quad G\bar{v} = \bar{\sigma}u, \quad Gu = \sigma u

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the “strongest”, “high-gain” or “most important” directions.
Example

(4.39) \[ G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \]

The singular value decomposition of \( G_1 \) is

\[ G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_\Sigma \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}}^{V^H} \]

The largest gain of 7.343 is for an input in the direction \( \bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix} \), the smallest gain of 0.272 is for an input in the direction \( \underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix} \). Since in (4.39) both inputs affect both outputs, we say that the system is interactive.
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The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*;

\[ \frac{\bar{\sigma}}{\sigma} = \frac{7.343}{0.272} = 27.0. \]

**Example**

**Shopping cart.** Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards, sideways and upwards. For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned.
Example: Distillation process. Steady-state model of a distillation column

\[ G = \begin{bmatrix}
87.8 & -86.4 \\
108.2 & -109.6
\end{bmatrix} \tag{4.40} \]

Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

\[ G = \begin{bmatrix}
0.625 & -0.781 \\
0.781 & 0.625
\end{bmatrix} \begin{bmatrix}
197.2 & 0 \\
0 & 1.39
\end{bmatrix} \begin{bmatrix}
0.707 & -0.708 \\
-0.708 & -0.707
\end{bmatrix}^H \]

\[ \underline{U} \underline{\Sigma} \underline{V}^H \tag{4.41} \]
The distillation process is ill-conditioned, and the condition number is $197.2/1.39 = 141.7$. For dynamic systems the singular values and their associated directions vary with frequency (Figure 6).

Figure 6: Typical plots of singular values
4.2.4  Singular values for performance [3.3.5]

Maximum singular value is very useful in terms of frequency-domain performance and robustness.

Performance measure for SISO systems:

\[
\left| \frac{e(\omega)}{r(\omega)} \right| = |S(j\omega)|
\]

Generalization for MIMO systems

\[
\|e(\omega)\|_2/\|r(\omega)\|_2
\]

(4.42)  \[
\sigma(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega))
\]
For performance we want the gain \( \|e(\omega)\|_2/\|r(\omega)\|_2 \) small for any direction of \( r(\omega) \)

\[
\bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \quad \forall \omega \quad \Rightarrow \quad \bar{\sigma}(w_PS) < 1, \quad \forall \omega
\]

(4.43)

\[
\begin{align*}
\text{where the } \mathcal{H}_\infty \text{ norm is defined as the peak of the maximum singular value of the frequency response}
\end{align*}
\]

(4.44)

\[
\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega))
\]
Typical singular values of $S(j\omega)$ in Figure 7.

Figure 7: Singular values of $S$ for a $2 \times 2$ plant with RHP-zero
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- Bandwidth, $\omega_B$: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

Since $S = (I + L)^{-1}$, the singular values inequality

$$\sigma(A) - 1 \leq \frac{1}{\bar{\sigma}(I+A)^{-1}} \leq \sigma(A) + 1$$

yields

(4.45)

$$\bar{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \sigma(L) + 1$$

- low $\omega$: $\sigma(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\sigma(L)}$

- high $\omega$: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$
4.3 Introduction to MIMO robustness [3.7]

4.3.1 Motivating robustness example no. 1: Spinning Satellite [3.7.1]

Angular velocity control of a satellite spinning about one of its principal axes:

\[
G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix}
    s - a^2 & a(s + 1) \\
    -a(s + 1) & s - a^2 
\end{bmatrix}; \quad a = 10
\]

A minimal, state-space realization, \( G = C(sI - A)^{-1}B + D \), is
Poles at $s = \pm ja$ For stabilization:

$$K = I$$

(4.48)\[ T(s) = GK(I + GK)^{-1} = \frac{1}{s + 1} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \]
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Nominal stability (NS). Two closed loop poles at $s = -1$ and

$$A_{cl} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nominal performance (NP). Figure 8(a)

![Graph](image)

Figure 8: Typical plots of singular values

(a) Spinning satellite in (4.46)

(b) Distillation process in (4.51)
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- $\sigma(L) \leq 1 \ \forall \omega$ poor performance in low gain direction
- $g_{12}, g_{21}$ large $\Rightarrow$ strong interaction

Robust stability (RS).
Check stability: one loop at a time.

Figure 9: Checking stability margins “one-loop-at-a-time”
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(4.49) \[ \frac{z_1}{w_1} \overset{\Delta}{=} L_1(s) = \frac{1}{s} \Rightarrow GM = \infty, PM = 90^\circ \]

- Good Robustness? NO
- Consider perturbation in each feedback channel

(4.50) \[ u'_1 = (1 + \epsilon_1)u_1, \quad u'_2 = (1 + \epsilon_2)u_2 \]

\[ B' = \begin{bmatrix} 1 + \epsilon_1 & 0 \\ 0 & 1 + \epsilon_2 \end{bmatrix} \]

Closed-loop state matrix:

\[ A'_{cl} = A - B'KC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 + \epsilon_1 & 0 \\ 0 & 1 + \epsilon_2 \end{bmatrix} \begin{bmatrix} 1 \\ -a \\ 1 \end{bmatrix} \]
Characteristic polynomial:

\[ \det(sI - A'_{cl}) = s^2 + (2 + \epsilon_1 + \epsilon_2) s + \]
\[ + a_1 + 1 + \epsilon_1 + \epsilon_2 + (a^2 + 1)\epsilon_1\epsilon_2 \]

Stability for \((-1 < \epsilon_1 < \infty, \epsilon_2 = 0)\) and \((\epsilon_1 = 0, -1 < \epsilon_2 < \infty)\) (GM=\(\infty\))

But only small simultaneous changes in the two channels: for example, let \(\epsilon_1 = -\epsilon_2\), then the system is unstable \((a_0 < 0)\) for

\[ |\epsilon_1| > \frac{1}{\sqrt{a^2 + 1}} \approx 0.1 \]

**Summary.** Checking single-loop margins is inadequate for MIMO problems.
4.3.2 Motivating robustness example no. 2: Distillation Process [3.7.2]

Idealized dynamic model of a distillation column,

\[
G(s) = \frac{1}{75s + 1} \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}
\]

(time is in minutes).
Figure 10: Response with decoupling controller to filtered reference input $r_1 = 1/(5s + 1)$. The perturbed plant has 20% gain uncertainty as given by (4.54).
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Inverse-based controller or equivalently steady-state decoupler with a PI controller ($k_1 = 0.7$)

\[(4.52) \quad K_{\text{inv}}(s) = \frac{k_1}{s} G^{-1}(s) = \frac{k_1(1 + 75s)}{s} \begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \]

Nominal performance (NP).

\[G K_{\text{inv}} = K_{\text{inv}} G = \frac{0.7}{s} I\]

first order response with time constant 1.43 (Fig. [10]).

Nominal performance (NP) achieved with decoupling controller.
Robust stability (RS).

\begin{equation}
S = S_I = \frac{s}{s + 0.7} \quad I; \quad T = T_I = \frac{1}{1.43s + 1} \quad I
\end{equation}

In each channel: GM=\(\infty\), PM=90°.

Input gain uncertainty (4.50) with \(\epsilon_1 = 0.2\) and \(\epsilon_2 = -0.2\):

\begin{equation}
u'_1 = 1.2u_1, \quad u'_2 = 0.8u_2
\end{equation}

\begin{equation}
L'_I(s) = K_{inv}G' = K_{inv}G \left[ \begin{array}{cc}
1 + \epsilon_1 & 0 \\
0 & 1 + \epsilon_2
\end{array} \right] =
\end{equation}

\begin{equation}
\frac{0.7}{s} \left[ \begin{array}{cc}
1 + \epsilon_1 & 0 \\
0 & 1 + \epsilon_2
\end{array} \right]
\end{equation}
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Perturbed closed-loop poles are

\begin{align*}
s_1 &= -0.7(1 + \epsilon_1), \\
s_2 &= -0.7(1 + \epsilon_2)
\end{align*}

Closed-loop stability as long as the input gains $1 + \epsilon_1$ and $1 + \epsilon_2$ remain positive

$\Rightarrow$ Robust stability (RS) achieved with respect to input gain errors for the decoupling controller.

Robust performance (RP).

Performance with model error poor (Fig. 10)

- SISO: NP+RS $\Rightarrow$ RP
- MIMO: NP+RS $\not\Rightarrow$ RP

RP is not achieved by the decoupling controller.
4.3.3 Robustness conclusions [3.7.3]

Multivariable plants can display a sensitivity to uncertainty (in this case input uncertainty) which is fundamentally different from what is possible in SISO systems.
4.4 General control problem formulation [3.8]

Figure 11: General control configuration for the case with no model uncertainty
The overall control objective is to minimize some norm of the transfer function from $w$ to $z$, for example, the $H_\infty$ norm. The controller design problem is then:

Find a controller $K$ which based on the information in $v$, generates a control signal $u$ which counteracts the influence of $w$ on $z$, thereby minimizing the closed-loop norm from $w$ to $z$. 
4.4.1 Obtaining the generalized plant $P$ [3.8.1]

The routines in MATLAB for synthesizing $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers assume that the problem is in the general form of Figure 11.

Example: One degree-of-freedom feedback control configuration.

![One degree-of-freedom control configuration](image)

Figure 12: One degree-of-freedom control configuration
Equivalent representation of Figure 12 where the error signal to be minimized is $z = y - r$ and the input to the controller is $v = r - y_m$.

Figure 13: General control configuration equivalent to Figure 12.
\( w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; z = e = y - r; v = r - y_m = r - y - n \)

\[
z = y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu
\]
\[
v = r - y_m = r - Gu - d - n = -Iw_1 + Iw_2 - Iw_3 - Gu
\]

and \( P \) which represents the transfer function matrix from \( [w \ u]^T \) to \( [z \ v]^T \) is

\[
(4.58) \quad P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix}
\]

Note that \( P \) does not contain the controller. Alternatively, \( P \) can be obtained from Figure 13.
Remark. In MATLAB we may obtain $P$ via `simulink`, or we may use the `sysic` program in the $\mu$-toolbox. The code in Table 1 generates the generalized plant $P$ in (4.58) for Figure 12.

Table 1: MATLAB program to generate $P$

```matlab
% Uses the Mu-toolbox
systemnames = 'G'; % G is the SISO plant.
inputvar = '[d(1);r(1);n(1);u(1)]'; % Consists of vectors $w$ and $u$.
input_to_G = '[u]';
outputvar = '[G+d-r; r-G-d-n]'; % Consists of vectors $z$ and $v$.
sysoutname = 'P';
sysic;
```
4.4.2 Including weights in $P$ [3.8.2]

To get a meaningful controller synthesis problem, for example, in terms of the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms, we generally have to include weights $W_z$ and $W_w$ in the generalized plant $P$, see Figure 14.

![Diagram of control system configuration](image)

Figure 14: General control configuration for the case with no model uncertainty
That is, we consider the weighted or normalized exogenous inputs $w$, and the weighted or normalized controlled outputs $z = W_z \tilde{z}$. The weighting matrices are usually frequency dependent and typically selected such that weighted signals $w$ and $z$ are of magnitude 1, that is, the norm from $w$ to $z$ should be less than 1.
Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem

\begin{align}
\underset{K}{\text{min}} \|N(K)\|_\infty, \quad N = \begin{bmatrix}
  W_uKS \\
  W_T T \\
  W_P S
\end{bmatrix}
\end{align}

where $K$ is a stabilizing controller. In other words, we have $z = Nw$ and the objective is to minimize the $\mathcal{H}_\infty$ norm from $w$ to $z$. 

Figure 15: Block diagram corresponding to generalized plant in (4.59)
so the generalized plant $P$ from $[w \ u]^T$ to $[z \ v]^T$ is

$$P = \begin{bmatrix}
0 & W_u I \\
0 & W_T G \\
W_P I & W_P G \\
-I & -G
\end{bmatrix}$$ (4.60)
4.4.3 Partitioning the generalized plant $P$ [3.8.3]

We often partition $P$ as

(4.61) \[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \]

so that

(4.62) \[ z = P_{11}w + P_{12}u \]

(4.63) \[ v = P_{21}w + P_{22}u \]
In Example “Stacked $S/T/KS$ problem” we get from (4.60)

\[
P_{11} = \begin{bmatrix} 0 \\ 0 \\ WPI \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_uI \\ W_TG \\ W_PG \end{bmatrix}
\]

(4.64)

\[
P_{21} = -I, \quad P_{22} = -G
\]

(4.65)

Note that $P_{22}$ has dimensions compatible with the controller $K$ in Figure 14.
4.4.4 Analysis: Closing the loop to get $N$ [3.8.4]

Figure 16: General block diagram for analysis with no uncertainty

For analysis of closed-loop performance we may absorb $K$ into the interconnection structure and obtain the system $N$ as shown in Figure 16 where

\begin{equation}
    z = Nw
\end{equation}
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where $N$ is a function of $K$. To find $N$, first partition the generalized plant $P$ as given in (4.61)-(4.63), combine this with the controller equation

(4.67) \[ u = Kv \]

and eliminate $u$ and $v$ from equations (4.62), (4.63) and (4.67) to yield $z = Nw$ where $N$ is given by

(4.68) \[ N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq F_l(P, K) \]

Here $F_l(P, K)$ denotes a lower linear fractional transformation (LFT) of $P$ with $K$ as the parameter. In words, $N$ is obtained from Figure 11 by using $K$ to close a lower feedback loop around $P$. Since positive feedback is used in the general configuration in Figure 11 the term $(I - P_{22}K)^{-1}$ has a negative sign.
Example: We want to derive $N$ for the partitioned $P$ in (4.64) and (4.65) using the LFT-formula in (4.68). We get

$$N = \begin{bmatrix} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \end{bmatrix} K (I + G K)^{-1} (-I) = \begin{bmatrix} -W_u K S \\ -W_T T \\ W_P S \end{bmatrix}$$

where we have made use of the identities $S = (I + G K)^{-1}$, $T = G K S$ and $I - T = S$.

In the MATLAB $\mu$-Toolbox we can evaluate $N = F_l(P, K)$ using the command $N = \text{starp}(P, K)$. Here $\text{starp}$ denotes the matrix star product which generalizes the use of LFTs.
4.4.5 Further examples [3.8.5]

Example: Consider the control system in Figure 17, where $y_1$ is the output we want to control, $y_2$ is a secondary output (extra measurement), and we also measure the disturbance $d$. The control configuration includes a two degrees-of-freedom controller, a feedforward controller and a local feedback controller based on the extra measurement $y_2$. 
Figure 17: System with feedforward, local feedback and two degrees-of-freedom control
To recast this into our standard configuration of Figure 11, we define

\[
w = \begin{bmatrix} d \\ r \end{bmatrix}; \quad z = y_1 - r; \quad v = \begin{bmatrix} r \\ y_1 \\ y_2 \\ d \end{bmatrix}
\]

\[(4.69)\]

\[
K = \begin{bmatrix} K_1 K_r & -K_1 & -K_2 & K_d \end{bmatrix}
\]

\[(4.70)\]

We get

\[
P = \begin{bmatrix} G_1 & -I & G_1 G_2 \\ 0 & I & 0 \\ G_1 & 0 & G_1 G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{bmatrix}
\]

\[(4.71)\]

Then partitioning \(P\) as in (4.62) and (4.63) yields

\[
P_{22} = \begin{bmatrix} 0^T & (G_1 G_2)^T & G_2^T & 0^T \end{bmatrix}^T.
\]
4.4.6 Deriving $P$ from $N$ [3.8.6]

For cases where $N$ is given and we wish to find a $P$ such that

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

it is usually best to work from a block diagram representation. This was illustrated above for the stacked $N$ in (4.59). Alternatively, the following procedure may be useful:

1. Set $K = 0$ in $N$ to obtain $P_{11}$.
2. Define $Q = N - P_{11}$ and rewrite $Q$ such that each term has a common factor $R = K(I - P_{22}K)^{-1}$ (this gives $P_{22}$).
3. Since $Q = P_{12}RP_{21}$, we can now usually obtain $P_{12}$ and $P_{21}$ by inspection.
Example 1  Weighted sensitivity. We will use the above procedure to derive $P$ when

$N = w_P S = w_P (I + GK)^{-1}$, 
where $w_P$ is a scalar weight.

1. $P_{11} = N(K = 0) = w_P I$.

2. $Q = N - w_P I = w_P (S - I) = -w_P T = -w_P GK (I + GK)^{-1}$, 
and we have $R = K (I + GK)^{-1}$ so $P_{22} = -G$.

3. $Q = -w_P G R$ so we have $P_{12} = -w_P G$ and $P_{21} = I$, and we get

$$P = \begin{bmatrix} w_P I & -w_P G \\ I & -G \end{bmatrix}$$ (4.72)
A general control configuration including model uncertainty [3.8.8]

The general control configuration in Figure 11 may be extended to include model uncertainty. Here the matrix $\Delta$ is a block-diagonal matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$. 
Figure 18: General control configuration for the case with model uncertainty
Figure 19: General block diagram for analysis with uncertainty included
Figure 20: Rearranging a system with multiple perturbations into the $N\Delta$-structure
The block diagram in Figure 18 in terms of $P$ (for synthesis) may be transformed into the block diagram in Figure 19 in terms of $N$ (for analysis) by using $K$ to close a lower loop around $P$. The same lower LFT as found in (4.68) applies, and

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

(4.73)

To evaluate the perturbed (uncertain) transfer function from external inputs $w$ to external outputs $z$, we use $\Delta$ to close the upper loop around $N$ (see Figure 19), resulting in an upper LFT:

$$z = F_u(N, \Delta)w; \quad F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

(4.74)
Remark 1 Almost any control problem with uncertainty can be represented by Figure 18. First represent each source of uncertainty by a perturbation block, $\Delta_i$, which is normalized such that $\|\Delta_i\| \leq 1$. Then “pull out” each of these blocks from the system so that an input and an output can be associated with each $\Delta_i$ as shown in Figure 20(a). Finally, collect these perturbation blocks into a large block-diagonal matrix having perturbation inputs and outputs as shown in Figure 20(b).