Multivariable Robust Control

Lecture 2 (Meetings 2-4)

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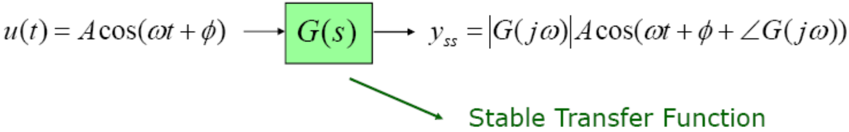
Frequency Response [2.1]

We use the *Frequency Response* to describe the response of the system to sinusoids of varying frequency.



Feedback control [2.2]

One dregree-of-freedom controller [2.2.1]

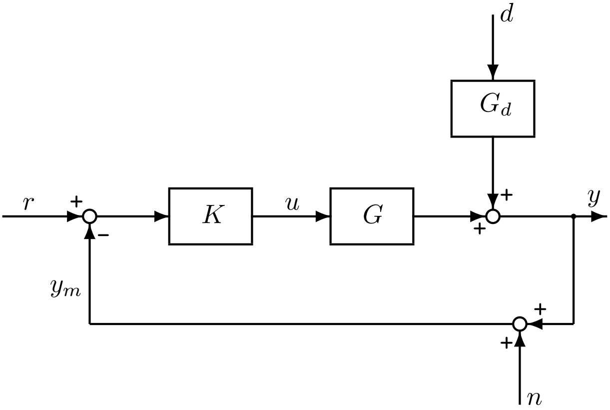


Figure 1: Block diagram of one degree-of-freedom feedback control system

Closed-loop transfer functions [2.2.2]

$$y = GK(r - y - n) + G_d d$$

or

$$(I + GK)y = GKr + G_d d - GKn \tag{2.1}$$

which implies that

$$y = \underbrace{(I + GK)^{-1}GK}_{T}\, r \tag{2.2}$$

$$+ \underbrace{(I + GK)^{-1}}_{S}\, G_d d \tag{2.3}$$

$$- \underbrace{(I + GK)^{-1}GK}_{T}\, n \tag{2.4}$$

Classical Feedback Control

Control error:

$$e = y - r = -Sr + SG_d d - Tn \tag{2.5}$$

Plant input:

$$u = KSr - KSG_d d - KSn \tag{2.6}$$

Note that:

$$L = GK \tag{2.7}$$

$$S = (I + GK)^{-1} = (I + L)^{-1} \tag{2.8}$$

$$T = (I + GK)^{-1} GK = (I + L)^{-1} L \tag{2.9}$$

Classical Feedback Control

Moreover:

$$S + T = I \tag{2.10}$$

Notation :

$L = GK$	loop transfer function
$S = (I+L)^{-1}$	sensitivity function
$T = (I+L)^{-1}L$	complementary sensitivity function

Closed-loop stability [2.3]

- Root-locus
- Routh-Hurwitz criterion
- Nyquist criterion

Example:

$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$

- The model has a right-half plane (RHP) zero at $s = 0.5$ rad/sec.
- The RHP zero imposes a fundamental limitation on control.

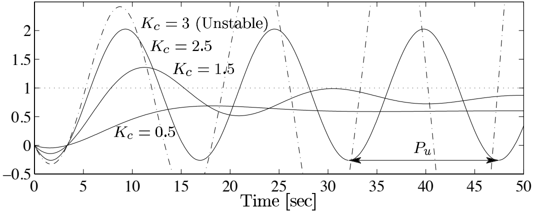


Figure 2: Effect of proportional gain K_c on closed-loop response $y(t)$.

Matlab: lecture02a_Inverse_PController_Stability.m

Closed-loop performance [2.4]

- Time-domain specifications
- Frequency-domain specifications

Example:

$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$

- The closed-loop system is stable for $-1/3 \leq K_c \leq 5/2$.
- The proportional gain cannot eliminate steady-state tracking error.

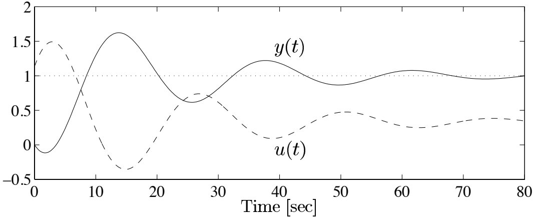


Figure 3: Closed-loop step response $y(t)$ with PI control.

Matlab: lecture02b_Inverse_PIController_Performance.m

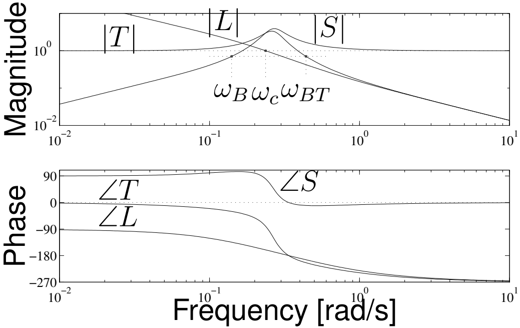


Figure 4: Bode magnitude and phase plots of $L = GK$, S and T when $G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$, and $K(s) = 1.136(1 + \frac{1}{12.7s})$.

Classical Feedback Control

Time domain performance [2.4.2]

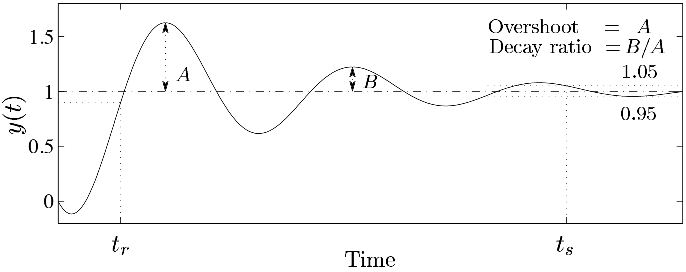


Figure 5: Step response analysis: Rise time (t_r), Settling time (t_s), Overshoot (A), Peak time (t_p), Decay Ratio (B), Steady-state Offset, Total Variation.

Classical Feedback Control

- Rise time (t_r): The time it takes for the output to first reach 90% of its final value, which is usually required to be small
- Settling time (t_s): The time after which the output remains within $\pm 5\%$ of its final value, which is usually required to be small
- Overshoot: The peak value divided by the final value, which should typically be less than 1.2 (20%)
- Decay ratio: The ratio of the second and first peaks, which should typically be less than 0.3
- Excess variation: The total variation (TV) divided by the overall change at steady state, which should be as close to 1 as possible

NOTE: For second-order systems there are analytical relationships between time specifications and location of the poles.

Frequency domain performance - Gain and phase margins [2.4.3]

If the Nyquist plot of L crosses the negative real axis between -1 and 0, the the (upper) gain margin (or gain amplification margin) is defined as

$$GM = 1/|L(j\omega_{180})| \tag{2.11}$$

where the phase crossover frequency ω_{180} is where the Nyquist curve of $L(j\omega_{180})$ crosses the negative real axis between -1 and 0, i.e.

$$\angle L(j\omega_{180}) = -180^o \tag{2.12}$$

If the Nyquist plot of L crosses the negative real axis between $-\infty$ and -1, the the (lower) gain margin (or gain reduction margin) is defined as

$$GM_L = 1/|L(j\omega_{L180})| \tag{2.13}$$

where the phase crossover frequency ω_{L180} is where the Nyquist curve of $L(j\omega_{180})$ crosses the negative real axis between $-\infty$ and -1.

Classical Feedback Control

The phase margin is defined as

$$PM = \angle L(j\omega_c) + 180^o \tag{2.14}$$

where the gain crossover frequency ω_c is the frequency where $|L(j\omega)|$ crosses 1, i.e.

$$|L(j\omega_c)| = 1 \tag{2.15}$$

Classical Feedback Control

Below, typical Nyquist plot of $L(j\omega)$ for stable plant. Closed-loop instability occurs if $L(j\omega)$ encircles -1.

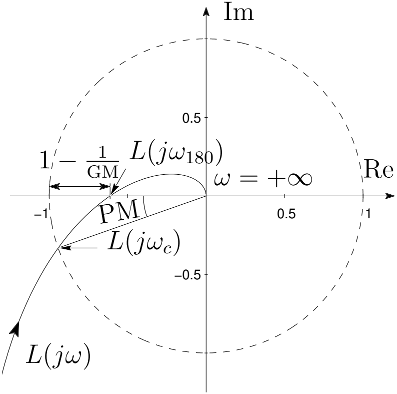


Figure 6: Typical Nyquist plot of $L(j\omega)$ for stable plant with PM and GM indicated. Closed-loop instability occurs if $L(j\omega)$ encircles the critical point -1.

Maximum peak criteria

Maximum peaks of sensitivity and complementary sensitivity functions:

$$M_S \stackrel{\triangle}{=} \max_{\omega} |S(j\omega)|; \quad M_T \stackrel{\triangle}{=} \max_{\omega} |T(j\omega)| \tag{2.16}$$

Typically :

$$M_S \leq 2 \quad (6dB) \tag{2.17}$$

$$M_T < 1.25 \quad (2dB) \tag{2.18}$$

Note :

$$\text{GM} \geq \frac{M_S}{M_S - 1} \tag{2.19}$$

$$\text{PM} \geq 2 \arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} \ [\text{rad}] \tag{2.20}$$

For example, for $M_S = 2$ we are guaranteed

$$\text{GM} \geq 2 \text{ and } \text{PM} \geq 29.0^\circ.$$

Bandwidth and crossover frequency

Bandwidth is defined as the frequency range $[\omega_1, \omega_2]$ over which control is "effective". Usually $\omega_1 = 0$, and then $\omega_2 = \omega_B$ is the bandwidth.

Definition: The (closed-loop) bandwidth, ω_B, is the frequency where $|S(j\omega)|$ first crosses $1/\sqrt{2} = 0.707(\approx -3$ dB) from below.

Definition: The bandwidth in terms of T, ω_{BT}, is the highest frequency at which $|T(j\omega)|$ crosses $1/\sqrt{2} = 0.707(\approx -3$ dB) from above. (Usually a poor indicator of performance).

The *gain crossover frequency*, ω_c, is the frequency where $|L(j\omega_c)|$ first crosses 1 from above. For systems with PM $< 90^\circ$ we have

$$\omega_B < \omega_c < \omega_{BT} \tag{2.21}$$

Classical Feedback Control

Controller design [2.5] Three main approaches:

1. **The transfer-function shaping approach.**
   1. **Loop shaping.** Classical approach in which the magnitude of the open-loop transfer function, $L(j\omega)$, is shaped. Usually no optimization is involved. Exception: $\mathcal{H}_\infty$ loop-shaping. Original loop-shaping is optimized in second step.
   2. **Shaping of closed-loop transfer functions, such as S, T and KS** Optimization is usually used $\Rightarrow$ $\mathcal{H}_\infty$ optimal control
2. **The signal-based approach.**
   One considers a particular disturbance or reference change and tries to optimize the closed-loop response $\Rightarrow$ Linear Quadratic Gaussian (LQG) control $\rightarrow$ $\mathcal{H}_2$-norm control (frequency dependent weights)
3. **The numerical optimization approach.**
   Multi-objective optimization to optimize directly the true objectives, such as rise times, stability margins, etc. Computationally difficult (particularly if the optimization problem is not convex). Optimization may be performed online $\Rightarrow$ Model Predictive Control (MPC).

Loop shaping [2.6]

Shaping of open loop transfer function $L(j\omega)$:

$$e = -\underbrace{(I+L)^{-1}}_{S} r + \underbrace{(I+L)^{-1}}_{S} G_d d - \underbrace{(I+L)^{-1} L}_{T} n \tag{2.22}$$

Fundamental trade-offs:

1. Good disturbance rejection: L large.
2. Good command following: L large.
3. Mitigation of measurement noise: L small.
4. Small magnitude of input signals: K small and L small.

NOTE: Fortunately, the conflicting design objectives are generally in different frequency ranges, and we can meet most of the objectives by using a large loop gain L at low frequencies and a small loop gain L at high frequencies above crossover.

Fundamentals of loop-shaping design

Specifications for desired loop transfer function:

1. Gain crossover frequency, ω_c, where $|L(j\omega_c)| = 1$.
2. The shape of $L(j\omega)$, e.g. slope of $|L(j\omega)|$ in certain frequency ranges: $N = \frac{d\,ln\,|L|}{d\,ln\,w}$ Typically, a slope $N = -1$ (-20 dB/decade) around crossover, and a larger roll-off at higher frequencies. The desired slope at lower frequencies depends on the nature of the disturbance or reference signal.
3. The system type, defined as the number of pure integrators in $L(s)$.

Note:

1. *To avoid tracking offset,* $L(s)$ *must contain at least one integrator for each integrator in* $r(s)$*.*
2. *Slope and phase are dependent. For example:* $\angle \frac{1}{s^n} = -n\,\frac{\pi}{2}$
3. *Design of* $L(s)$ *is most crucial around* ω_c *and* ω_{180} *due to stability constraints.*

Classical Feedback Control

Example:
$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$

Chosen loop gain:
$L(s) = \frac{3K_c(-2s+1)}{s(2s+1)(0.33s+1)}$

- The RHP zero cannot be cancelled by the controller. Therefore, L must contain the RHP zero of G.
- The RHP zero imposes a performance limitation. The crossover frequency must be $\omega_c < 0.5\omega_z$. In this case, $\omega_z = 0.5$.
- We require the system to have one integrator (type 1 system)

CONCLUSION: L must have a slope of $-20dB/dec$ at low frequencies and then roll off with a higher slope at frequencies beyond ω_z.

Inverse-based controller [2.6.3]

Note: $L(s)$ must contain all RHP-zeros of $G(s)$. Idea for minimum-phase plant:

$$L(s) = \frac{\omega_c}{s} \tag{2.23}$$

$$K(s) = \frac{\omega_c}{s} G^{-1}(s) \tag{2.24}$$

i.e. controller inverts plant and adds integrator $(1/s)$. Note that the controller will not be realizable if $G(s)$ has more poles than zeros.

BUT: A slope of $N = -1$ at all frequencies is *not* generally desirable, unless references and disturbances affect the outputs as steps. This is illustrated by the following example.

Example: Disturbance process.

$$\boxed{G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}, \quad G_d(s) = \frac{100}{10s+1}} \tag{2.25}$$

Objectives are:

1. Command tracking: rise time (to reach 90% of the final value) less than 0.3 s and overshoot less than 5%.
2. Disturbance rejection: response to unit step disturbance should stay within the range $[-1, 1]$ at all times, and should return to 0 as quickly as possible ($|y(t)|$ should at least be less than 0.1 after 3 s).
3. Input constraints: $u(t)$ should remain within $[-1, 1]$

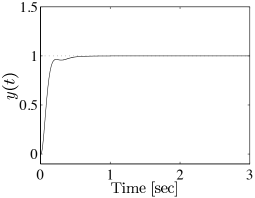
Analysis. $|G_d(j\omega)|$ remains larger than 1 up to $\omega_d \approx 10$ rad/s $\Rightarrow$ $\underline{\omega_c \approx 10 \text{ rad/s}}$.

Note: We do not want ω_c higher than necessary because of stability to noise and stability issues associated with high-gain feedback.

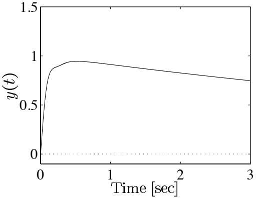
Inverse-based controller design.

$$\begin{aligned} K_0(s) &= \frac{\omega_c}{s}\frac{10s+1}{200}(0.05s+1)^2 \\ &\approx \frac{\omega_c}{s}\frac{10s+1}{200}\frac{0.1s+1}{0.01s+1}, \end{aligned}$$

$$L_0(s) = \frac{\omega_c}{s}\frac{0.1s+1}{(0.05s+1)^2(0.01s+1)},\ \omega_c = 10 \qquad (2.26)$$



(a) Tracking



(b) Disturbance

Figure 7: Responses with "inverse-based" controller $K_0(s)$ for the disturbance process. Note excellent tracking response but poor disturbance-rejection response.

Loop shaping for disturbance rejection [2.6.4]

$$e = y = SG_d d, \tag{2.27}$$

to achieve $|e(\omega)| \leq 1$ for $|d(\omega)| = 1$ (the worst-case disturbance) we require $|SG_d(j\omega)| < 1, \forall\omega$, or

$$|1 + L| \geq |G_d| \quad \forall\omega \tag{2.28}$$

or approximately

$$|L| \geq |G_d| \quad \forall\omega \tag{2.29}$$

Initial guess:

$$|L_{\min}| \approx |G_d| \tag{2.30}$$

or

$$|K_{\min}| \approx |G^{-1}G_d| \tag{2.31}$$

Classical Feedback Control

Controller contains the model of the disturbance.

$$|K_{\min}| \approx |G^{-1}G_d| \tag{2.32}$$

Summary:

- For disturbance rejection, a good choice is a controller that contains the dynamics (G_d) of the disturbance and inverts the dynamics (G) of the inputs (at least at frequencies just before crossover).
- For disturbance at plant output, $G_d = 1$, and we get $|K_{\min}| = |G^{-1}|$. Then, an inverse-based controller provides best trade-off between performance (disturbance rejection) and minimum use of feedback.
- For disturbances at plant input we have $G_d = G$, and we get $|K_{\min}| = 1$. Then, a constant unitary controller offers good trade-off between output performance and input usage.

Classical Feedback Control

In addition to satisfying $|L| \approx |G_d|$ at frequencies around crossover, the desired loop-shape $L(s)$ may be modified as follows:

1. Around crossover make slope N of $|L|$ to be about -1 for transient behaviour with acceptable gain and phase margins.
2. Increase the loop gain at low frequencies to improve the settling time and reduce the steady-state offset $\rightarrow$ add an integrator $\rightarrow$ add zero to reduce phase lag $\rightarrow$ add gain $k > 1$ to speed up the response.

$$|K| = k\left|\frac{s+\omega_I}{s}\right| \, |\underbrace{G^{-1}G_d}_{K_{min}}| \tag{2.33}$$

3. Let $L(s)$ roll off faster at higher frequencies (beyond the bandwidth) in order to reduce the use of manipulated inputs, to make the controller realizable and to reduce the effects of noise.
4. $L(s)$ must be selected such closed-loop system is stable

Example: Loop-shaping design for the disturbance process.

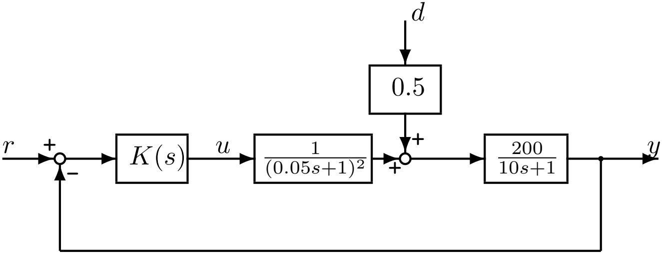


Figure 8: Block diagram representation of the disturbance process in (2.25)

Step 1. Initial design. $K(s) = G^{-1}G_d = 0.5(0.05s+1)^2$.
Make proper:

$$K_1(s) = 0.5 \tag{2.34}$$

$\Longrightarrow$ offset!
Step 2. More gain at low frequency. To get integral action multiply the controller by the term $\frac{s+\omega_I}{s}$. For $\omega_I = 0.2\omega_c$ the phase contribution from $\frac{s+\omega_I}{s}$ is $\arctan(1/0.2) - 90^\circ = -11^\circ$ at ω_c. For $\omega_c \approx 10$ rad/s, select the following controller

$$K_2(s) = 0.5\frac{s+2}{s} \tag{2.35}$$

$\Longrightarrow$ response exceeds 1, oscillatory, small phase margin

Step 3. High-frequency correction. Supplement with "derivative action" by multiplying $K_2(s)$ by a lead-lag term effective over one decade starting at 20 rad/s:

$$K_3(s) = 0.5\frac{s+2}{s}\;\frac{0.05s+1}{0.005s+1} \tag{2.36}$$

$\Longrightarrow$ poor reference tracking (simulation)

NOTE: For reference tracking we want the controller to look like $\frac{G^{-1}}{s}$, while for disturbance rejection we want the controller to look like $\frac{G^{-1}G_d}{s}$. We cannot achieve both goals with a single feedback controller.

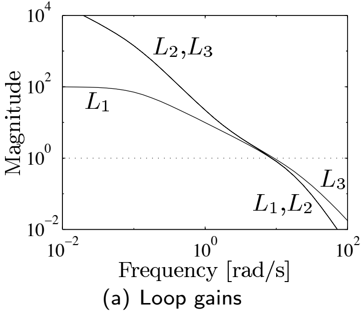


Figure 9: Loop shapes and disturbance responses for controllers K_1, K_2 and K_3 for the disturbance process.

	GM	PM	ω_c	M_S	M_T	Reference		Disturbance	
						t_r	$y_{\max}$	$y_{\max}$	$y(t=3)$
Spec.$\rightarrow$			≈ 10			$\leq .3$	≤ 1.05	≤ 1	≤ 0.1
K_0	9.95	72.9°	11.4	1.34	1	.16	1.00	0.95	.75
K_1	4.04	44.7°	8.48	1.83	1.33	.21	1.24	1.35	.99
K_2	3.24	30.9°	8.65	2.28	1.89	.19	1.51	1.27	.001
K_3	19.7	50.9°	9.27	1.43	1.23	.16	1.24	0.99	.001

Figure 10: Alternative loop-shaping designs for the disturbance process.

*Two degrees of freedom design [2.6.5]

Reference tracking: $K \sim \frac{G^{-1}}{s}$
Disturbance rejection: $K \sim \frac{G^{-1}G_d}{s}$

In order to meet both regulator and tracking performance use K_r (= "prefilter"):

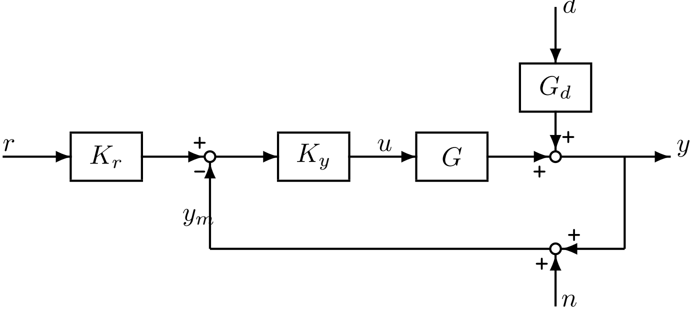


Figure 11: Two degrees-of-freedom controller

Idea:

- Design K_y
- $T = L(I+L)^{-1}$ with $L = GK_y$
- Desired $y = T_{ref} r$

$$\Longrightarrow K_r = T^{-1} T_{ref} \tag{2.37}$$

Remark:
Practical choice of prefilter is the lead-lag network

$$K_r(s) = \frac{\tau_{\text{lead}} s + 1}{\tau_{\text{lag}} s + 1} \tag{2.38}$$

$\tau_{\text{lead}} > \tau_{\text{lag}}$ to speed up the response, and $\tau_{\text{lead}} < \tau_{\text{lag}}$ to slow down the response.

Example: Two degrees-of-freedom design for the disturbance process.

Adopt $K_y = K_3$. Approximate response by inpection of y_3:
$T(s) \approx \frac{1.5}{0.1s+1} - \frac{0.5}{0.5s+1} = \frac{(0.7s+1)}{(0.1s+1)(0.5s+1)}$

Adopt $T_r(s) = \frac{1}{(0.1s+1)}$, which yields:
$K_r(s) = \frac{0.5s+1}{0.7s+1}$.

By closed-loop simulations, we slightly modify controller:

$$K_{r3}(s) = \frac{0.5s+1}{0.65s+1} \cdot \frac{1}{0.03s+1} \tag{2.39}$$

where $1/(0.03s+1)$ included to avoid initial peaking of input signal $u(t)$ above 1.

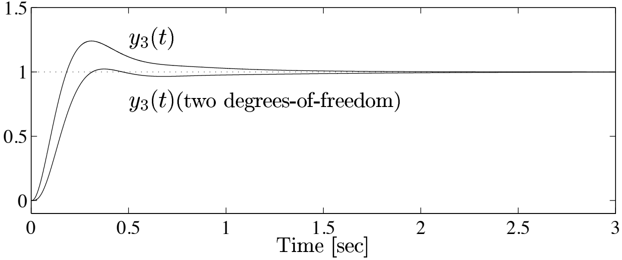


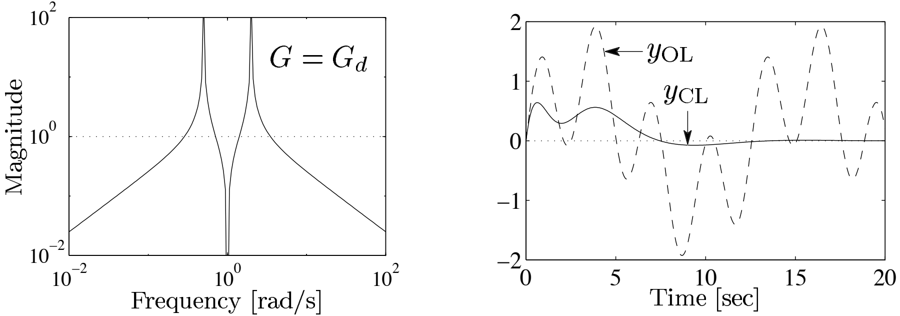
Figure 12: Tracking responses with the one degree-of-freedom controller (K_3) and the two degrees-of-freedom controller (K_3, K_{r3}) for the disturbance process

Example: Loop shaping for a flexible structure.

$$G(s) = G_d(s) = \frac{2.5s(s^2+1)}{(s^2+0.5^2)(s^2+2^2)} \qquad (2.40)$$

$|K_{\min}(j\omega)| = |G^{-1}G_d| = 1 \Rightarrow$

$$K(s) = 1 \qquad (2.41)$$



(a) Magnitude plot of $|G| = |G_d|$ (b) OL & CL disturbance responses with $K = 1$

Figure 13: Flexible structure in (2.40)

Closed-loop shaping [2.8]

Why ?

We are interested in S and T:

$$\begin{array}{lll} |L(j\omega)| \gg 1 & \Rightarrow & S \approx L^{-1}; \quad T \approx 1 \\ |L(j\omega)| \ll 1 & \Rightarrow & S \approx 1; \qquad T \approx L \end{array}$$

but in the crossover region where $|L(j\omega)|$ is close to 1, one cannot infer anything about S and T from $|L(j\omega)|$.

Alternative:
Directly shape the magnitudes of closed-loop $S(s)$ and $T(s)$.

The term $\mathcal{H}_\infty$

The $\mathcal{H}_\infty$ norm of a stable scalar transfer function $f(s)$ is simply the peak value of $|f(j\omega)|$ as a function of frequency, that is,

$$\|f(s)\|_\infty \triangleq \max_\omega |f(j\omega)| \tag{2.42}$$

The symbol ∞ comes from:

$$\max_\omega |f(j\omega)| = \lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |f(j\omega)|^p d\omega \right)^{1/p}$$

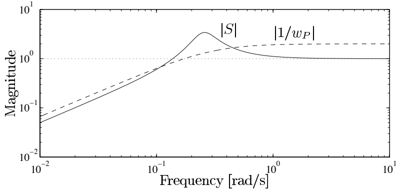
The symbol $\mathcal{H}$ stands for "Hardy space", and $\mathcal{H}_\infty$ is the set of transfer functions with bounded ∞-norm, which is simply the set of *stable and proper* transfer functions.

Weighted sensitivity [2.8.2]

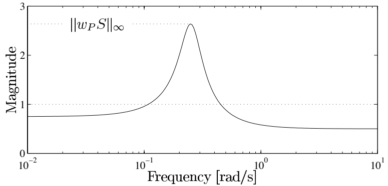
Typical specifications in terms of S:

1. Minimum bandwidth frequency ω_B^*.
2. Maximum tracking error at selected frequencies.
3. System type, or alternatively the maximum steady-state tracking error, A.
4. Shape of S over selected frequency ranges.
5. Maximum peak magnitude of S, $\|S(j\omega)\|_\infty \leq M$.

Specifications may be captured by an upper bound, $1/|w_P(s)|$, on $\|S\|$.



(a) Sensitivity S and performance weight w_P



(b) Weighted sensitivity $w_P S$

Figure 14: $|S|$ exceeds its bound $1/|w_P| \Rightarrow \|w_P S\|_\infty > 1$

$$|S(j\omega)| < 1/|w_P(j\omega)|, \ \forall\omega \tag{2.43}$$

$$\Leftrightarrow \quad |w_P S| < 1, \ \forall\omega \quad \Leftrightarrow \quad \boxed{\|w_P S\|_\infty < 1} \tag{2.44}$$

Typical performance weight:

$$\boxed{w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}} \tag{2.45}$$

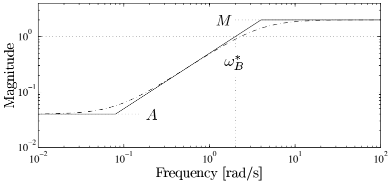


Figure 15: Inverse of performance weight. Exact/asymptotic plot of $1/|w_P(j\omega)|$ in (2.45)

Classical Feedback Control

To get a steeper slope for L (and S) below the bandwidth:

$$\boxed{w_P(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}} \tag{2.46}$$

***Mixed sensitivity [2.8.3]**

In order to enforce specifications on other transfer functions:

$$\|N\|_\infty = \max_\omega \bar{\sigma}(N(j\omega)) < 1; \quad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u KS \end{bmatrix} N = \begin{bmatrix} w_P S \\ w_T T \\ w_u KS \end{bmatrix} \tag{2.47}$$

N is a vector and the maximun singular value $\bar{\sigma}(N)$ is the usual Euclidean vector norm:

$$\bar{\sigma}(N) = \sqrt{|w_P S|^2 + |w_T T|^2 + |w_u KS|^2} \tag{2.48}$$

The $\mathcal{H}_\infty$ optimal controller is obtained from

$$\min_K \|N(K)\|_\infty \tag{2.49}$$

Example: $\mathcal{H}_\infty$ mixed sensitivity design for the disturbance process.

Consider the plant in (2.25), and an $\mathcal{H}_\infty$ mixed sensitivity S/KS design in which

$$N = \begin{bmatrix} w_P S \\ w_u KS \end{bmatrix} \tag{2.50}$$

Selected $w_u = 1$ and

$$w_{P1}(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}; \quad M = 1.5,\ \omega_B^* = 10, \quad A = 10^{-4} \tag{2.51}$$

$\Longrightarrow$ poor disturbance response

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To get higher gains at low frequencies:

$$w_{P2}(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2} \tag{2.52}$$

with $M = 1.5, \omega_B^* = 10, A = 10^{-4}$.

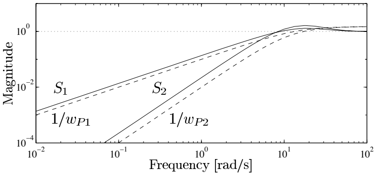
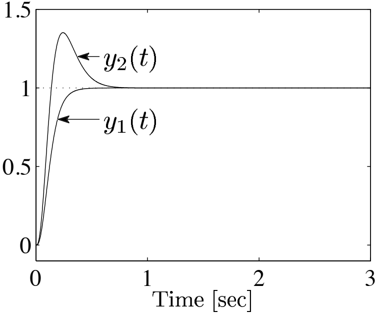


Figure 16: Inverse of performance weight (dashed line) and resulting sensitivity function (solid line) for two $\mathcal{H}_\infty$ designs (1 and 2) for the disturbance process



(a) Tracking response

(b) Disturbance response

Figure 17: Closed-loop step responses for two alternative $\mathcal{H}_\infty$ designs (1 and 2) for the disturbance process