# NONLINEAR BOUNDARY CONTROL OF NON-BURNING PLASMA IN CYLINDRICAL GEOMETRY

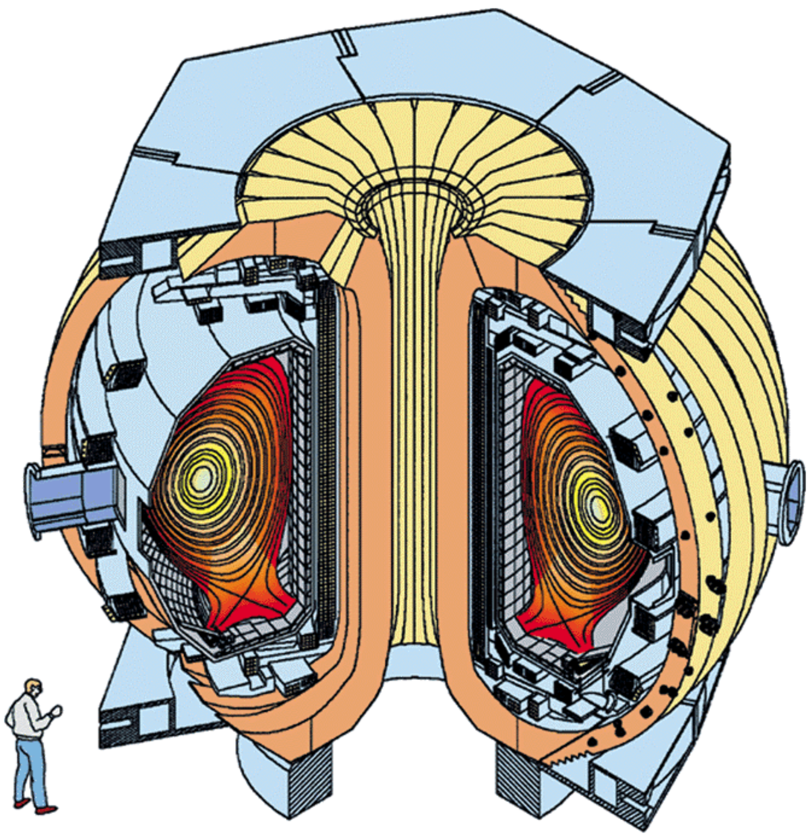

*Eugenio Schuster - Miroslav Krstic*

UCSD

*University of California San Diego*

*Department of Mechanical and Aerospace Engineering*

*Dynamic Systems and Control Group*



## Kinetic Control

The goal of the controller is to make the kinetic (density and temperature) radial profiles converge to their desired equilibrium profiles.

We are interested in constructing a stabilizing controller that:

- ✓ achieves stability for unstable equilibrium profiles
- ✓ increases performance for stable equilibrium profiles

***Why do we want to control the kinetic profiles?***

Burn Control
MHD Instability Avoidance
High-beta and High-confinement modes access Confinement Time Improvement
Transport Reduction

# Model – Non-Burning plasma

**Energy Equation**

$$\frac{3}{2}\frac{\partial nT}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(k\frac{\partial T}{\partial r}+D\frac{3}{2}T\frac{\partial n}{\partial r}\right)\right]-P_{brem}+P_{aux}$$

$$\chi=\frac{n(0)}{n(r)}\frac{m^2}{\sec},\qquad k=n\chi=n(0)\frac{m^2}{\sec},\qquad D=\frac{2}{3}\chi=\frac{2}{3}\frac{n(0)}{n(r)}\frac{m^2}{\sec},\qquad V_p=0.5\frac{D}{T}\frac{\partial T}{\partial r}$$

$$\frac{\partial E}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial E}{\partial r}\right]-P_{brem}+P_{aux}$$

**Density Balance Equation**

$$\frac{\partial n}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(D\frac{\partial n}{\partial r}-nV_p\right)\right]+S$$

$$P_{brem}=A_bZ_{eff}n_e^2\sqrt{T},\qquad Z_{eff}=\left(\sum_i n_iZ_i^2\right)\Big/n_e=1$$

**Boundary Conditions:** $\frac{\partial E}{\partial r}(0,t)=0,\quad \frac{\partial E}{\partial r}(a,t)=k_EE(a,t),\quad \frac{\partial n}{\partial r}(0,t)=0,\quad \frac{\partial n}{\partial r}(a,t)=k_nn(a,t)$

# Equilibrium Profiles

$$\boxed{\begin{aligned} E(r,t) &= \overline{E}(r) + \widetilde{E}(r,t) \\ n(r,t) &= \overline{n}(r) + \widetilde{n}(r,t) \end{aligned}}$$

**Equilibrium Equations**

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[ r\overline{D}\frac{\partial \overline{E}}{\partial r}\right] - \overline{P}_{brem} + \overline{P}_{aux}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[ r\left(\overline{D}\frac{\partial \overline{n}}{\partial r} - \overline{n}\,\overline{V}_p\right)\right] + \overline{S}$$

**Boundary Conditions**

$$\frac{\partial \overline{E}}{\partial r}(0) = 0, \qquad \frac{\partial \overline{E}}{\partial r}(a) = k_E \overline{E}(a), \qquad \frac{\partial \overline{n}}{\partial r}(0) = 0, \qquad \frac{\partial \overline{n}}{\partial r}(a) = k_n \overline{n}(a)$$

# Deviation Equations - Energy

**Energy Equation:**

$$\frac{\partial \tilde{E}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[ rD \frac{\partial\left(\bar{E}+\tilde{E}\right)}{\partial r}\right] - P_{brem} + P_{aux}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[ rD \frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[ rD \frac{\partial \bar{E}}{\partial r}\right] - P_{brem} + P_{aux}$$

$$= \frac{\partial}{\partial r}\left[ D \frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r} D \frac{\partial \tilde{E}}{\partial r} + g\left(E,n\right)$$

$$g\left(E,n\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[ rD \frac{\partial \bar{E}}{\partial r}\right] - P_{brem} + P_{aux}$$

# Deviation Equations - Density

**Density Equation:**

$$\frac{\partial \tilde{n}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[ r\left( D\frac{\partial(\bar{n}+\tilde{n})}{\partial r} - n\left(\bar{V}_p^* + \tilde{V}_p^*\right)\right)\right] + S$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[ r\left( D\frac{\partial \tilde{n}}{\partial r} - n\tilde{V}_p^*\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[ r\left( D\frac{\partial \bar{n}}{\partial r} - n\bar{V}_p^*\right)\right] + S$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[ rD\left( \frac{3}{2}\frac{\partial \tilde{n}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{E}}{\partial r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[ rD\left( \frac{3}{2}\frac{\partial \bar{n}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \bar{E}}{\partial r}\right)\right] + S$$

$$= \frac{3}{2}\left\{ \frac{\partial}{\partial r}\left[ D\frac{\partial \tilde{n}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{n}}{\partial r}\right\} - \frac{1}{2}\left\{ \frac{\partial}{\partial r}\left[ \frac{Dn}{E}\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \tilde{E}}{\partial r}\right\} + f(E,n)$$

$$f(E,n) = \frac{3}{2}\left\{ \frac{\partial}{\partial r}\left[ D\frac{\partial \bar{n}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \bar{n}}{\partial r}\right\} - \frac{1}{2}\left\{ \frac{\partial}{\partial r}\left[ \frac{Dn}{E}\frac{\partial \bar{E}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \bar{E}}{\partial r}\right\}$$

$$V_p = \frac{1}{2}\frac{D}{T}\frac{\partial T}{\partial r} = \frac{1}{2}\frac{D}{(2E/3n)}\frac{\partial(2E/3n)}{\partial r} = \frac{1}{2}\frac{Dn}{E}\frac{n\frac{\partial E}{\partial r} - E\frac{\partial n}{\partial r}}{n^2} = \frac{1}{2}D\left(\frac{1}{E}\frac{\partial E}{\partial r} - \frac{1}{n}\frac{\partial n}{\partial r}\right)$$

$$V_p = \frac{1}{2}D\left(\frac{1}{E}\frac{\partial \bar{E}}{\partial r} - \frac{1}{n}\frac{\partial \bar{n}}{\partial r}\right) + \frac{1}{2}D\left(\frac{1}{E}\frac{\partial \tilde{E}}{\partial r} - \frac{1}{n}\frac{\partial \tilde{n}}{\partial r}\right) = \bar{V}_p^* + \tilde{V}_p^*$$

**Boundary Conditions:**

$$\frac{\partial \tilde{E}}{\partial r}(0,t) = 0, \qquad \frac{\partial \tilde{E}}{\partial r}(a,t) = k_E \tilde{E}(a,t) + \Delta \tilde{E}_r$$

$$\frac{\partial \tilde{n}}{\partial r}(0,t) = 0, \qquad \frac{\partial \tilde{n}}{\partial r}(a,t) = k_n \tilde{n}(a,t) + \Delta \tilde{n}_r$$

## Deviation Equations

**Energy Equation:**

$$\frac{\partial \tilde{E}}{\partial t} = \frac{\partial}{\partial r}\left[D\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{E}}{\partial r} + g(E,n)$$

$$g(E,n) = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \overline{E}}{\partial r}\right] - P_{brem} + P_{aux}$$

**Density Equation:**

$$\frac{\partial \tilde{n}}{\partial t} = \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial \tilde{n}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{n}}{\partial r}\right\} - \frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \tilde{E}}{\partial r}\right\} + f(E,n)$$

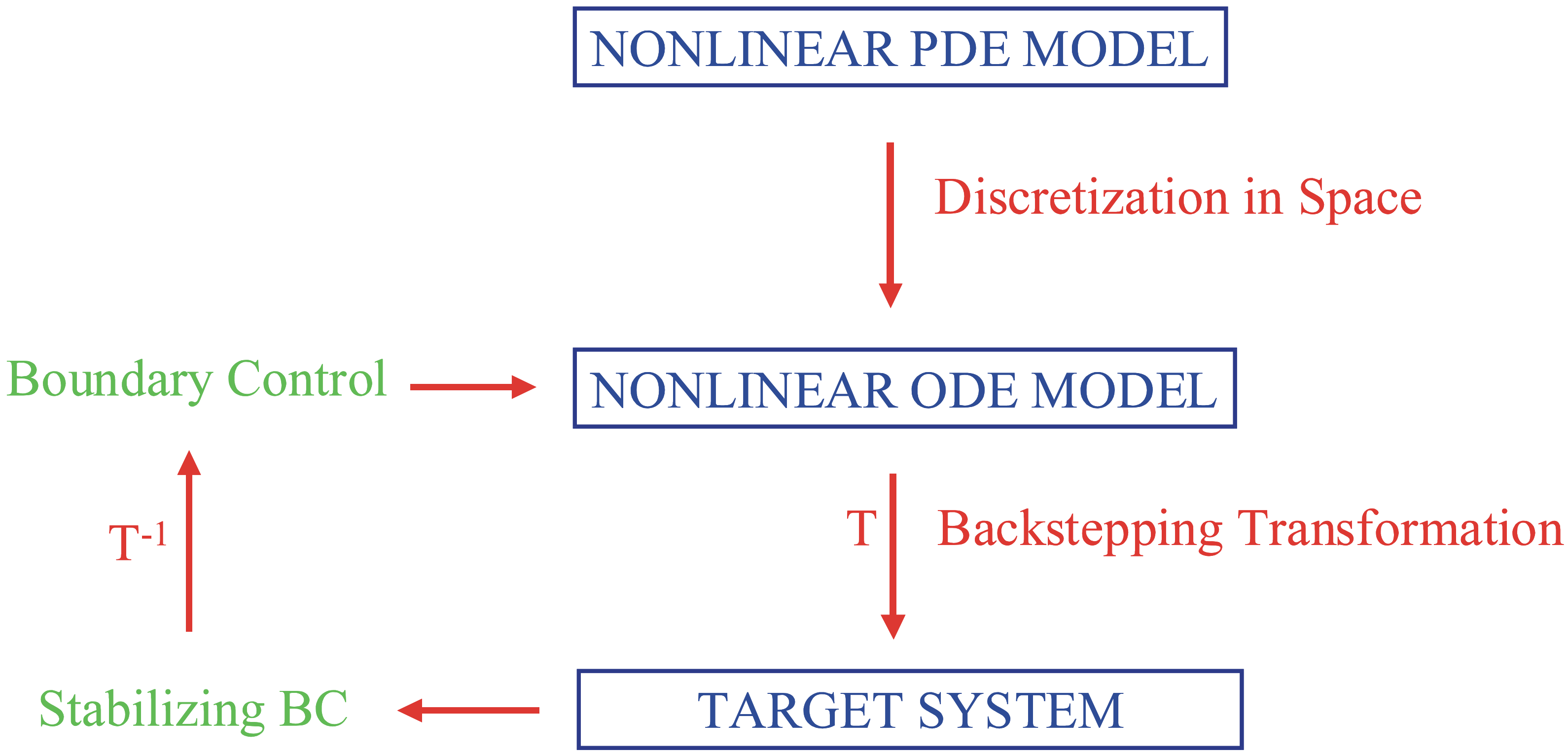
$$f(E,n) = \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial \overline{n}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \overline{n}}{\partial r}\right\} - \frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial \overline{E}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \overline{E}}{\partial r}\right\} + S$$

**Boundary Conditions:**

$$\frac{\partial \tilde{E}}{\partial r}(0,t) = 0, \qquad \frac{\partial \tilde{E}}{\partial r}(a,t) = k_E \tilde{E}(a,t) + \Delta \tilde{E}_r$$

$$\frac{\partial \tilde{n}}{\partial r}(0,t) = 0, \qquad \frac{\partial \tilde{n}}{\partial r}(a,t) = k_n \tilde{n}(a,t) + \Delta \tilde{n}_r$$

# Controller Design – Method Summary



# Controller Design – Discretization Method

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rH\frac{\partial x}{\partial r}\right]=\frac{\partial}{\partial r}\left[H\frac{\partial x}{\partial r}\right]+\frac{1}{r}H\frac{\partial x}{\partial r}$$

$$\left.\frac{\partial x}{\partial r}\right|_i=\frac{x_{i+1}-x_i}{h}$$

$$\frac{\partial}{\partial r}\left[H\frac{\partial x}{\partial r}\right]_i=\frac{H_{i+1/2}\left.\frac{\partial x}{\partial r}\right|_{i+1/2}-H_{i-1/2}\left.\frac{\partial x}{\partial r}\right|_{i-1/2}}{h}$$

$$=\frac{H_{i+1/2}\left(\frac{x_{i+1}-x_i}{h}\right)-H_{i-1/2}\left(\frac{x_i-x_{i-1}}{h}\right)}{h}$$

$$=\frac{H_{i+1/2}x_{i+1}-2H_i x_i-H_{i-1/2}x_{i-1}}{h^2}$$

$$H_{i-1/2}=H_i-\frac{H_i-H_{i-1}}{h}\frac{h}{2}=\frac{1}{2}H_i+\frac{1}{2}H_{i-1},\qquad H_{i+1/2}=H_i+\frac{H_i-H_{i-1}}{h}\frac{h}{2}=\frac{3}{2}H_i-\frac{1}{2}H_{i-1}$$

**Energy Equation:**

$$\dot{\tilde{E}}_i = \frac{D_{i+1/2}\tilde{E}_{i+1} - 2D_i\tilde{E}_i + D_{i-1/2}\tilde{E}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{E}_{i+1} - \tilde{E}_i}{h} + g_i$$

$$g_i = \frac{D_{i+1/2}\bar{E}_{i+1} - 2D_i\bar{E}_i + D_{i-1/2}\bar{E}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\bar{E}_{i+1} - \bar{E}_i}{h} - \left(P_{brem}\right)_i + \left(P_{aux}\right)_i$$

$$\left(P_{brem}\right)_i = A_b n_i^2\sqrt{\frac{2}{3}\frac{E_i}{n_i}}$$

$$D_{i-1/2} = D_i - \frac{D_i - D_{i-1}}{h}\frac{h}{2} = \frac{1}{2}D_i + \frac{1}{2}D_{i-1}$$

$$D_{i+1/2} = D_i + \frac{D_i - D_{i-1}}{h}\frac{h}{2} = \frac{3}{2}D_i - \frac{1}{2}D_{i-1}$$

**Boundary Conditions:**

$$\frac{\tilde{E}_1 - \tilde{E}_0}{h} = 0, \qquad \frac{\tilde{E}_N - \tilde{E}_{N-1}}{h} = k_E\tilde{E}_N + \Delta\tilde{E}_r$$

**Density Equation:**

$$\dot{\tilde{n}}_i = \frac{3}{2}\left\{\frac{D_{i+1/2}\tilde{n}_{i+1} - 2D_i\tilde{n}_i + D_{i-1/2}\tilde{n}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{n}_{i+1} - \tilde{n}_i}{h}\right\}$$

$$-\frac{1}{2}\left\{\frac{(Dn/E)_{i+1/2}\tilde{E}_{i+1} - 2(Dn/E)_i\tilde{E}_i + (Dn/E)_{i-1/2}\tilde{E}_{i-1}}{h^2} + \frac{1}{ih}(Dn/E)_i\frac{\tilde{E}_{i+1} - \tilde{E}_i}{h}\right\} + f_i$$

$$f_i = \frac{3}{2}\left\{\frac{D_{i+1/2}\bar{n}_{i+1} - 2D_i\bar{n}_i + D_{i-1/2}\bar{n}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\bar{n}_{i+1} - \bar{n}_i}{h}\right\}$$

$$-\frac{1}{2}\left\{\frac{(Dn/E)_{i+1/2}\bar{E}_{i+1} - 2(Dn/E)_i\bar{E}_i + (Dn/E)_{i-1/2}\bar{E}_{i-1}}{h^2} + \frac{1}{ih}(Dn/E)_i\frac{\bar{E}_{i+1} - \bar{E}_i}{h}\right\} + S_i$$

$$(Dn/E)_{i-1/2} = (Dn/E)_i - \frac{(Dn/E)_i - (Dn/E)_{i-1}}{h}\frac{h}{2} = \frac{1}{2}(Dn/E)_i + \frac{1}{2}(Dn/E)_{i-1}$$

$$(Dn/E)_{i+1/2} = (Dn/E)_i + \frac{(Dn/E)_i - (Dn/E)_{i-1}}{h}\frac{h}{2} = \frac{3}{2}(Dn/E)_i - \frac{1}{2}(Dn/E)_{i-1}$$

**Boundary Conditions:**

$$\frac{\tilde{n}_1 - \tilde{n}_0}{h} = 0, \qquad \frac{\tilde{n}_N - \tilde{n}_{N-1}}{h} = k_E\tilde{n}_N + \Delta\tilde{n}_r$$

# Equilibrium Profiles

**Energy Equation:**

$$0=\frac{\overline{D}_{i+1/2}\overline{E}_{i+1}-2\overline{D}_i\overline{E}_i+\overline{D}_{i-1/2}\overline{E}_{i-1}}{h^2}+\frac{1}{ih}\overline{D}_i\frac{\overline{E}_{i+1}-\overline{E}_i}{h}-\left(\overline{P}_{brem}\right)_i+\left(\overline{P}_{aux}\right)_i$$

**Density Equation:**

$$0=\frac{3}{2}\left\{\frac{\overline{D}_{i+1/2}\overline{n}_{i+1}-2\overline{D}_i\overline{n}_i+\overline{D}_{i-1/2}\overline{n}_{i-1}}{h^2}+\frac{1}{ih}\overline{D}_i\frac{\overline{n}_{i+1}-\overline{n}_i}{h}\right\}$$

$$-\frac{1}{2}\left\{\frac{\left(\overline{D}\overline{n}/\overline{E}\right)_{i+1/2}\overline{E}_{i+1}-2\left(\overline{D}\overline{n}/\overline{E}\right)_i\overline{E}_i+\left(\overline{D}\overline{n}/\overline{E}\right)_{i-1/2}\overline{E}_{i-1}}{h^2}+\frac{1}{ih}\left(\overline{D}\overline{n}/\overline{E}\right)_i\frac{\overline{E}_{i+1}-\overline{E}_i}{h}\right\}+\overline{S}_i$$

**Boundary Conditions:**

$$\frac{\overline{E}_1-\overline{E}_0}{h}=0,\qquad \frac{\overline{E}_N-\overline{E}_{N-1}}{h}=k_E\overline{E}_N,\qquad \frac{\overline{n}_1-\overline{n}_0}{h}=0,\qquad \frac{\overline{n}_N-\overline{n}_{N-1}}{h}=k_E\overline{n}_N$$

**Pseudo-Energy Equation:**

$$\frac{\partial \tilde{F}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C_F\tilde{F}$$

$$= \frac{\partial}{\partial r}\left[D\frac{\partial \tilde{F}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{F}}{\partial r} - C_F\tilde{F}$$

**Pseudo-Density Equation:**

$$\frac{\partial \tilde{m}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\right] - C_m\tilde{m}$$

$$= \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial \tilde{m}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{m}}{\partial r}\right\} - \frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial \tilde{F}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \tilde{F}}{\partial r}\right\} - C_m\tilde{m}$$

**Boundary Conditions:**

$$\frac{\partial \tilde{F}}{\partial r}(0,t) = 0, \qquad \frac{\partial \tilde{F}}{\partial r}(a,t) = ?, \qquad \frac{\partial \tilde{m}}{\partial r}(0,t) = 0, \qquad \frac{\partial \tilde{m}}{\partial r}(a,t) = ?$$

**Pseudo-Energy Equation:**

$$\dot{\tilde{F}}_i = \frac{D_{i+1/2}\tilde{F}_{i+1} - 2D_i\tilde{F}_i + D_{i-1/2}\tilde{F}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h} - C_F\tilde{F}_i$$

**Pseudo-Density Equation:**

$$\dot{\tilde{m}}_i = \frac{3}{2}\left\{\frac{D_{i+1/2}\tilde{m}_{i+1} - 2D_i\tilde{m}_i + D_{i-1/2}\tilde{m}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{m}_{i+1} - \tilde{m}_i}{h}\right\}$$

$$-\frac{1}{2}\left\{\frac{(Dn/E)_{i+1/2}\tilde{F}_{i+1} - 2(Dn/E)_i\tilde{F}_i + (Dn/E)_{i-1/2}\tilde{F}_{i-1}}{h^2} + \frac{1}{ih}(Dn/E)_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h}\right\} - C_m\tilde{m}_i$$

**Boundary Conditions:**

$$\frac{\tilde{F}_1 - \tilde{F}_0}{h} = 0, \quad \frac{\tilde{F}_N - \tilde{F}_{N-1}}{h} = -G\tilde{F}_N, \quad \frac{\tilde{m}_1 - \tilde{m}_0}{h} = 0, \quad \frac{\tilde{m}_N - \tilde{m}_{N-1}}{h} = -G\tilde{m}_N, \quad G > 0$$

# Controller Design – Target System - Stability

$$\boxed{V=\frac{1}{2}\int_0^a r\left\{\frac{\widetilde{F}^2}{k^2}+\widetilde{m}^2\right\}dr}$$

$$\dot{V}=\frac{1}{2}\int_0^a r\left\{\frac{\widetilde{F}}{k^2}\dot{\widetilde{F}}+\widetilde{m}^2\dot{\widetilde{m}}\right\}dr$$

$$=\frac{1}{2}\int_0^a r\left\{\frac{\widetilde{F}}{k^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\widetilde{F}}{\partial r}\right]-C_F\widetilde{F}\right)+\widetilde{m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial\widetilde{m}}{\partial r}-\frac{1}{2}\frac{n}{E}\frac{\partial\widetilde{F}}{\partial r}\right)\right]-C_m\widetilde{m}\right)\right\}dr$$

$$=\frac{\widetilde{F}}{k^2}rD\frac{\partial\widetilde{F}}{\partial r}\bigg|_0^a-\int_0^a\frac{1}{k^2}rD\left(\frac{\partial\widetilde{F}}{\partial r}\right)^2dr-\frac{C_F}{k^2}\int_0^a r\widetilde{F}^2\,dr$$

$$+\widetilde{m}rD\left(\frac{3}{2}\frac{\partial\widetilde{m}}{\partial r}-\frac{1}{2}\frac{n}{E}\frac{\partial\widetilde{F}}{\partial r}\right)\bigg|_0^a-\int_0^a\frac{\partial\widetilde{m}}{\partial r}rD\left(\frac{3}{2}\frac{\partial\widetilde{m}}{\partial r}-\frac{1}{2}\frac{n}{E}\frac{\partial\widetilde{F}}{\partial r}\right)dr-C_m\int_0^a r\widetilde{m}^2\,dr$$

$$=aD(a)\left[\frac{\widetilde{F}(a)}{k^2}\widetilde{F}_r(a)+\frac{3}{2}\widetilde{m}(a)\widetilde{m}_r(a)-\frac{1}{2}\frac{n(a)}{E(a)}\widetilde{m}(a)\widetilde{F}_r(a)\right]$$

$$-\int_0^a r\left(\frac{C_F}{k^2}\widetilde{F}^2+C_m\widetilde{m}^2\right)dr-\frac{3}{2}\int_0^a rD\widetilde{m}_r^2\,dr-\int_0^a r\frac{D}{k^2}\widetilde{F}_r^2\,dr+\frac{1}{3}\int_0^a rD\frac{\widetilde{F}_r\widetilde{m}_r}{T}\,dr$$

$$\boxed{\tilde{F}_r(a) = -G\tilde{F}(a) \qquad \tilde{m}_r(a) = -G\tilde{m}(a)}$$

**Stabilizing Boundary Conditions**

$$\dot{V} = -GaD(a)\left[\frac{\tilde{F}^2(a)}{k^2} + \frac{3}{2}\tilde{m}^2(a) - \frac{1}{2}\tilde{m}(a)\tilde{F}(a)\frac{n(a)}{E(a)}\right]$$

$$-\int_0^a r\left(\frac{C_F}{k^2}\tilde{F}^2 + C_m\tilde{m}^2\right)dr - \frac{3}{2}\int_0^a rD\tilde{m}_r^2\,dr - \int_0^a r\frac{D}{k^2}\tilde{F}_r^2\,dr + \frac{1}{3}\int_0^a rD\frac{\tilde{F}_r\tilde{m}_r}{T}\,dr$$

$$= -GaD(a)\left[\frac{\tilde{F}^2(a)}{k^2} + \frac{3}{2}\tilde{m}^2(a) - \frac{1}{2}\tilde{m}(a)\tilde{F}(a)\frac{n(a)}{3/2\,n(a)kT^*(a)}\right]$$

$$-\int_0^a r\left(\frac{C_F}{k^2}\tilde{F}^2 + C_m\tilde{m}^2\right)dr - \frac{1}{2}\int_0^a rD\tilde{m}_r^2\,dr - \int_0^a r\frac{D}{k^2}\tilde{F}_r^2\,dr - \int_0^a rD\tilde{m}_r^2\,dr + \frac{1}{3}\int_0^a rD\frac{\tilde{F}_r\tilde{m}_r}{kT^*}\,dr$$

Considering that $T^* >> 1$ and defining $C = \min(C_F, C_m)$ we have

$$\dot{V} \le -CV - GaD(a)\left[\frac{\tilde{F}^2(a)}{k^2} + \tilde{m}^2(a) - \left|\frac{\tilde{F}(a)}{k}\right|\left|\tilde{m}(a)\right|\right] - \int_0^a rD\left[\frac{\tilde{F}_r^2}{k^2} + \tilde{m}_r^2 - \left|\frac{\tilde{F}_r}{k}\right|\left|\tilde{m}_r\right|\right]dr$$

By Young’s Inequality we conclude that

$$\dot{V} \le -CV$$

# Controller Design – Backstepping Transformation

$$\boxed{\begin{aligned}\tilde{F}_i &= \tilde{E}_i - \alpha_{i-1}\left(\tilde{E}_1,\ldots,\tilde{E}_{i-1},\tilde{n}_1,\ldots,\tilde{n}_{i-1}\right)\\ \tilde{m}_i &= \tilde{n}_i - \beta_{i-1}\left(\tilde{E}_1,\ldots,\tilde{E}_{i-1},\tilde{n}_1,\ldots,\tilde{n}_{i-1}\right)\end{aligned}}$$

$$\dot{\tilde{E}}_i = \frac{D_{i+1/2}\tilde{E}_{i+1} - 2D_i\tilde{E}_i + D_{i-1/2}\tilde{E}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{E}_{i+1}-\tilde{E}_i}{h} + g_i$$

$$\dot{\tilde{F}}_i = \frac{D_{i+1/2}\tilde{F}_{i+1} - 2D_i\tilde{F}_i + D_{i-1/2}\tilde{F}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{F}_{i+1}-\tilde{F}_i}{h} - C_F\tilde{F}_i$$

$$\boxed{\alpha_i = \frac{1}{D_{i+1/2}+D_i/i}\left[\left(2D_i + \frac{D_i}{i} + C_F h^2\right)\alpha_{i-1} - D_{i-1/2}\alpha_{i-2} - h^2 g_i - h^2 C_F\tilde{E}_i + h^2\dot{\alpha}_{i-1}\right]}$$

$$\dot{\tilde{n}}_i = \frac{3}{2}\left\{\frac{D_{i+1/2}\tilde{n}_{i+1} - 2D_i\tilde{n}_i + D_{i-1/2}\tilde{n}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{n}_{i+1}-\tilde{n}_i}{h}\right\} - \frac{1}{2}\left\{\frac{(Dn/E)_{i+1/2}\tilde{E}_{i+1} - 2(Dn/E)_i\tilde{E}_i + (Dn/E)_{i-1/2}\tilde{E}_{i-1}}{h^2} + \frac{1}{ih}(Dn/E)_i\frac{\tilde{E}_{i+1}-\tilde{E}_i}{h}\right\} + f_i$$

$$\dot{\tilde{m}}_i = \frac{3}{2}\left\{\frac{D_{i+1/2}\tilde{m}_{i+1} - 2D_i\tilde{m}_i + D_{i-1/2}\tilde{m}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{m}_{i+1}-\tilde{m}_i}{h}\right\} - \frac{1}{2}\left\{\frac{(Dn/E)_{i+1/2}\tilde{F}_{i+1} - 2(Dn/E)_i\tilde{F}_i + (Dn/E)_{i-1/2}\tilde{F}_{i-1}}{h^2} + \frac{1}{ih}(Dn/E)_i\frac{\tilde{F}_{i+1}-\tilde{F}_i}{h}\right\} - C_m\tilde{m}_i$$

$$\boxed{\begin{aligned}\beta_i &= \frac{1}{3/2\left(D_{i+1/2}+D_i/i\right)}\left[\left(\frac{3}{2}\left(2D_i + \frac{D_i}{i}\right) + C_m h^2\right)\beta_{i-1} - \frac{3}{2}D_{i-1/2}\beta_{i-2} - h^2 f_i + r_i - h^2 C_m\tilde{n}_i + h^2\dot{\beta}_{i-1}\right]\\ r_i &= \frac{1}{2}\left\{\left(\frac{Dn}{E}\right)_{i+1/2}\alpha_i - 2\left(\frac{Dn}{E}\right)_i\alpha_{i-1} + \left(\frac{Dn}{E}\right)_{i-1/2}\alpha_{i-2} + \frac{1}{i}\left(\frac{Dn}{E}\right)_i\left(\alpha_i - \alpha_{i-1}\right)\right\}\end{aligned}}$$

**Boundary Conditions for the Original System:**

$$\frac{\tilde{E}_1 - \tilde{E}_0}{h} = 0, \qquad \frac{\tilde{E}_N - \tilde{E}_{N-1}}{h} = k_E \tilde{E}_N + \Delta \tilde{E}_r, \qquad \frac{\tilde{n}_1 - \tilde{n}_0}{h} = 0, \qquad \frac{\tilde{n}_N - \tilde{n}_{N-1}}{h} = k_E \tilde{n}_N + \Delta \tilde{n}_r$$

**Boundary Conditions for the Target System:**

$$\frac{\tilde{F}_1 - \tilde{F}_0}{h} = 0, \qquad \frac{\tilde{F}_N - \tilde{F}_{N-1}}{h} = -G \tilde{F}_N, \qquad \frac{\tilde{m}_1 - \tilde{m}_0}{h} = 0, \qquad \frac{\tilde{m}_N - \tilde{m}_{N-1}}{h} = -G \tilde{m}_N$$

$$\Delta \tilde{E}_r = \frac{\alpha_{N-1} - \alpha_{N-2}}{h} - k_E \tilde{E}_N - G\left(\tilde{E}_N - \alpha_{N-1}\right)$$

$$\Delta \tilde{n}_r = \frac{\beta_{N-1} - \beta_{N-2}}{h} - k_n \tilde{n}_N - G\left(\tilde{n}_N - \beta_{N-1}\right)$$

$$\boxed{\begin{aligned} \tilde{E}_N &= \alpha_{N-1} + \frac{1}{(1+Gh)}\left(\tilde{E}_{N-1} - \alpha_{N-2}\right) \\ \tilde{n}_N &= \beta_{N-1} + \frac{1}{(1+Gh)}\left(\tilde{n}_{N-1} - \beta_{N-2}\right) \end{aligned}}$$

$$\widetilde{E}_i^{n+1} = \widetilde{E}_i^n + \Delta t \left\{ \frac{D^n_{i+1/2}\widetilde{E}^n_{i+1} - 2D^n_i \widetilde{E}^n_i + D^n_{i-1/2}\widetilde{E}^n_{i-1}}{h^2} + \frac{1}{ih} D^n_i \frac{\widetilde{E}^n_{i+1}-\widetilde{E}^n_i}{h} + g^n_i \right\}$$

$$g_i^n = \frac{D^n_{i+1/2}\overline{E}_{i+1} - 2D^n_i\overline{E}_i + D^n_{i-1/2}\overline{E}_{i-1}}{h^2} + \frac{1}{ih}D^n_i\frac{\overline{E}_{i+1}-\overline{E}_i}{h} - \left(P_{brem}\right)^n_i + \left(P_{aux}\right)_i$$

$$\left(P_{brem}\right)^n_i = A_b \left(n^n_i\right)^2 \sqrt{\frac{2}{3}\frac{E^n_i}{n^n_i}}$$

$$\widetilde{n}_i^{n+1} = \widetilde{n}_i^n + \Delta t \left\{ \frac{3}{2}\left[ \frac{D^n_{i+1/2}\widetilde{n}^n_{i+1} - 2D^n_i \widetilde{n}^n_i + D^n_{i-1/2}\widetilde{n}^n_{i-1}}{h^2} + \frac{1}{ih} D^n_i \frac{\widetilde{n}^n_{i+1}-\widetilde{n}^n_i}{h} \right] \right.$$

$$\left. - \frac{1}{2}\left[ \frac{\left(Dn/E\right)^n_{i+1/2}\widetilde{E}^n_{i+1} - 2\left(Dn/E\right)^n_i \widetilde{E}^n_i + \left(Dn/E\right)^n_{i-1/2}\widetilde{E}^n_{i-1}}{h^2} + \frac{1}{ih}\left(Dn/E\right)^n_i \frac{\widetilde{E}^n_{i+1}-\widetilde{E}^n_i}{h} \right] + f^n_i \right\}$$

$$f_i^n = \frac{3}{2}\left\{ \frac{D^n_{i+1/2}\overline{n}^n_{i+1} - 2D^n_i \overline{n}^n_i + D^n_{i-1/2}\overline{n}^n_{i-1}}{h^2} + \frac{1}{ih} D^n_i \frac{\overline{n}^n_{i+1}-\overline{n}^n_i}{h} \right\}$$

$$- \frac{1}{2}\left\{ \frac{\left(Dn/E\right)^n_{i+1/2}\overline{E}^n_{i+1} - 2\left(Dn/E\right)^n_i \overline{E}^n_i + \left(Dn/E\right)^n_{i-1/2}\overline{E}^n_{i-1}}{h^2} + \frac{1}{ih}\left(Dn/E\right)^n_i \frac{\overline{E}^n_{i+1}-\overline{E}^n_i}{h} \right\} + S_i$$

# Simulation Results

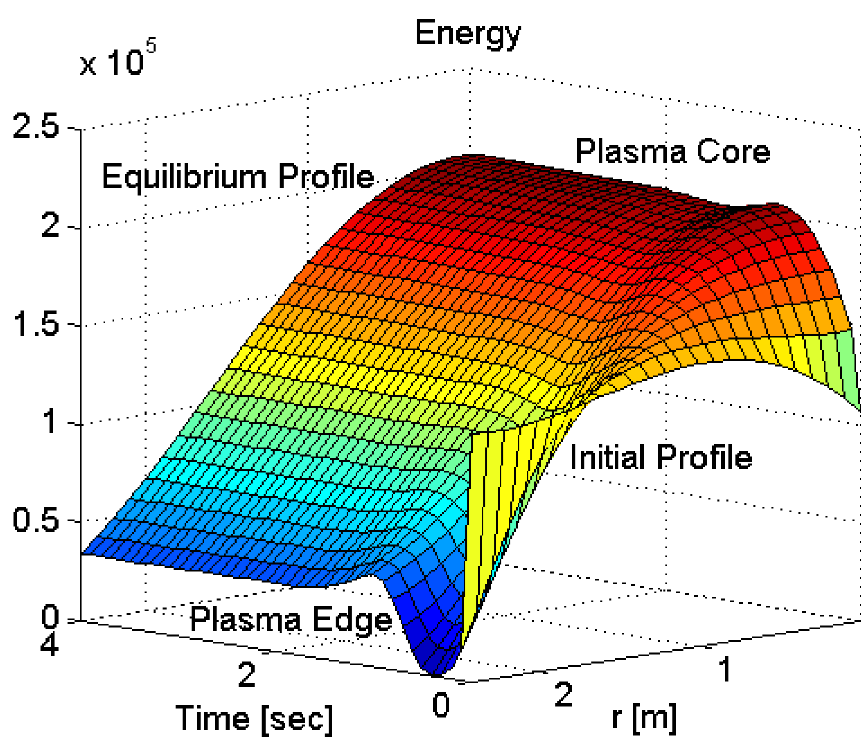



Figure 1-a: Energy Profile Evolution

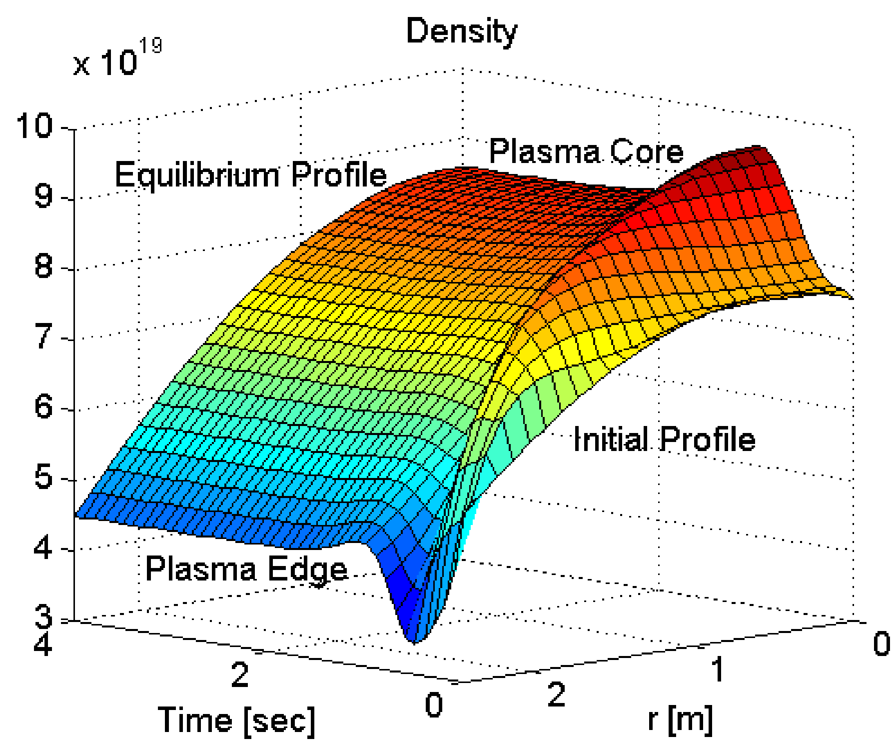
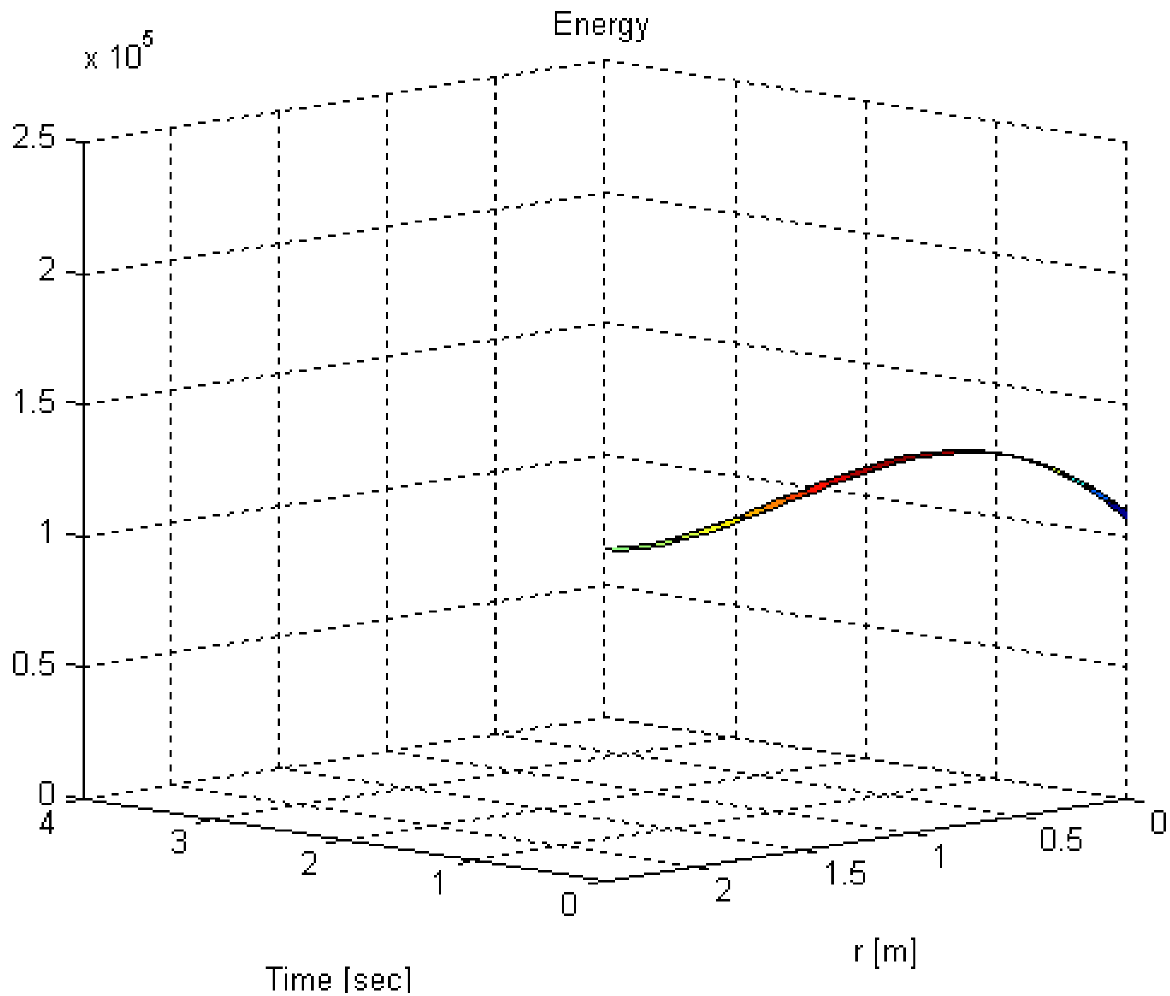


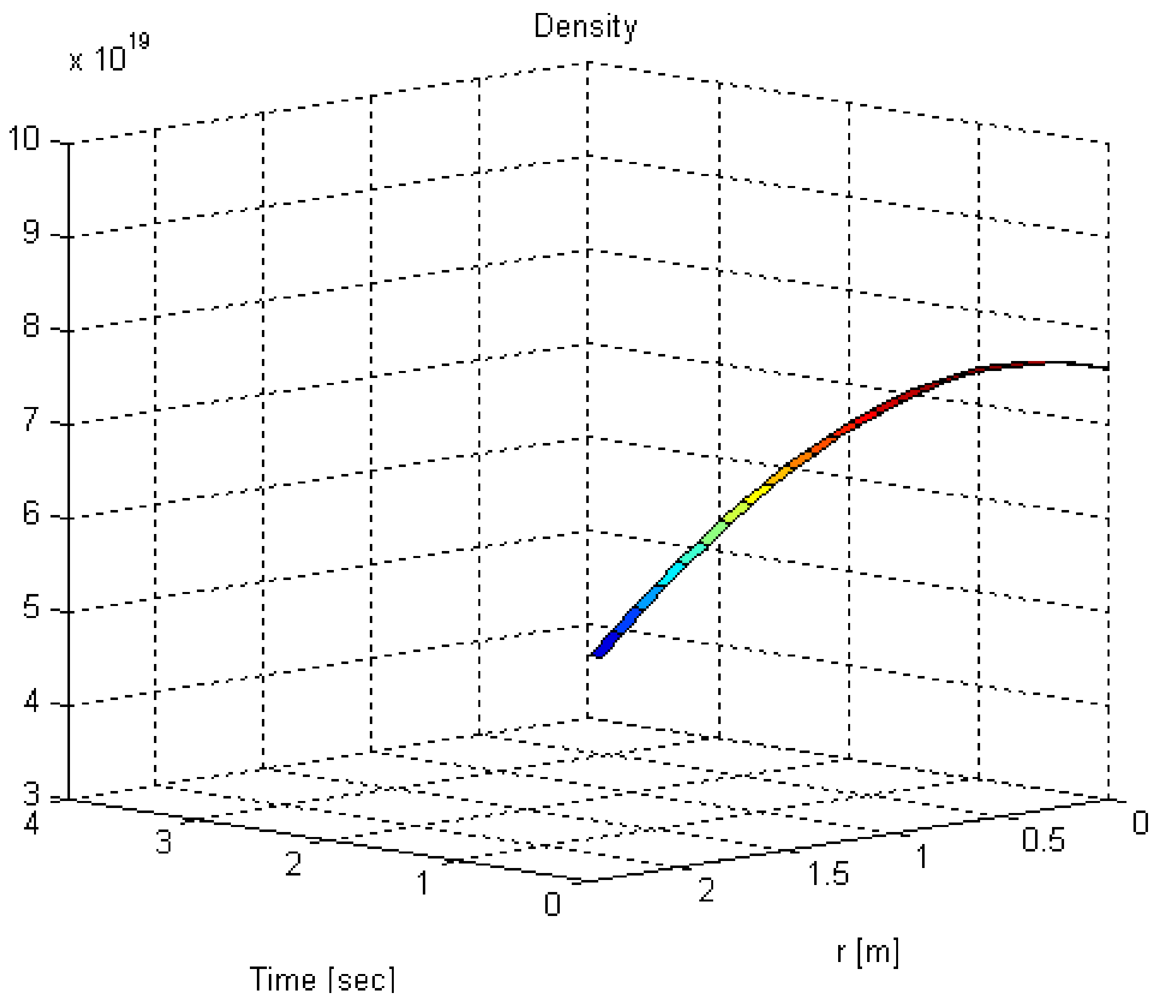

Figure 1-b: Density Profile Evolution

# Simulation Results



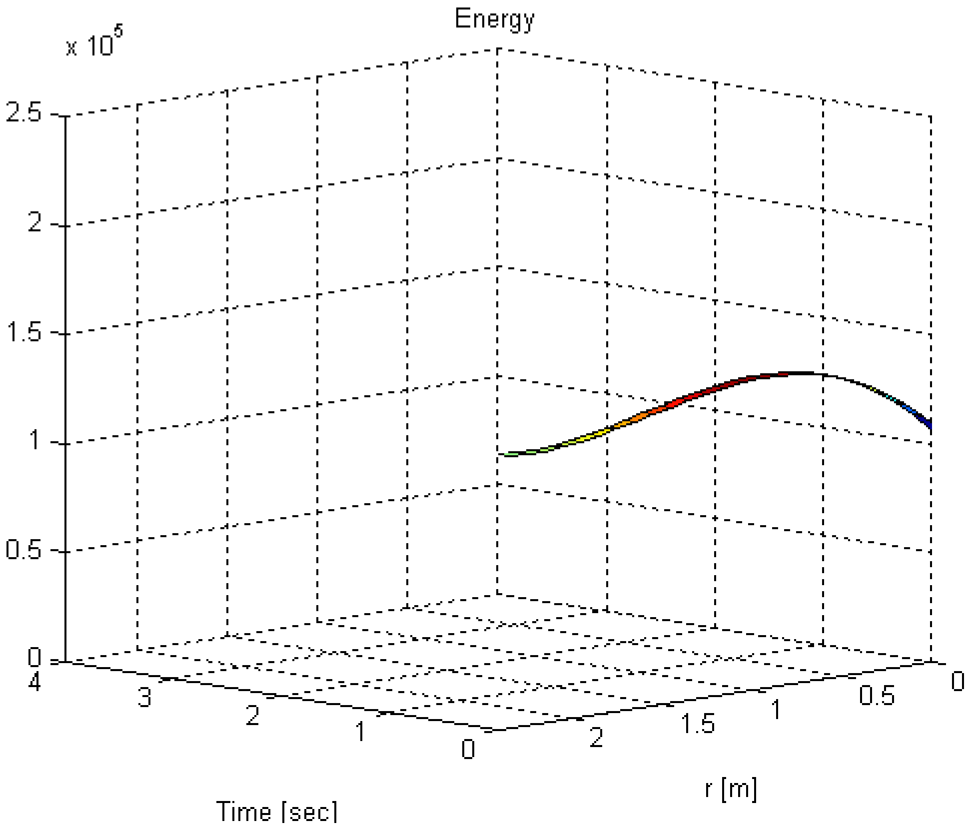

Energy Profile Evolution

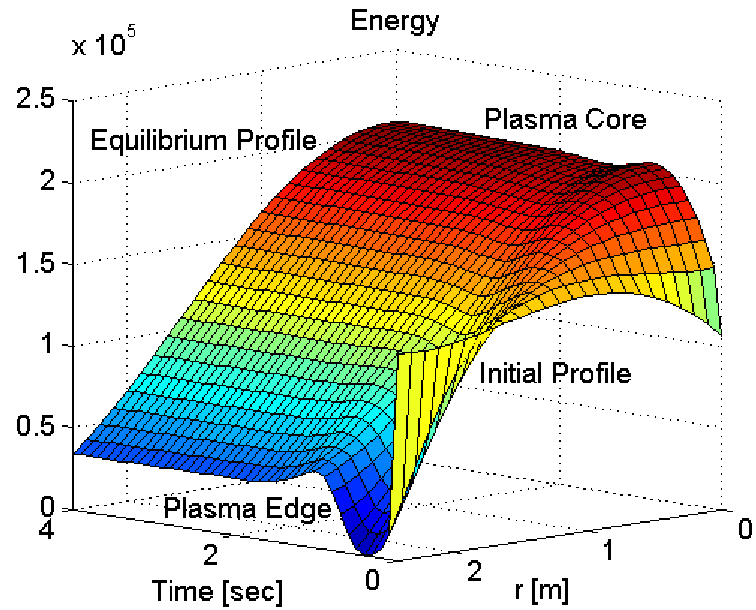
## Simulation Results




Density Profile Evolution

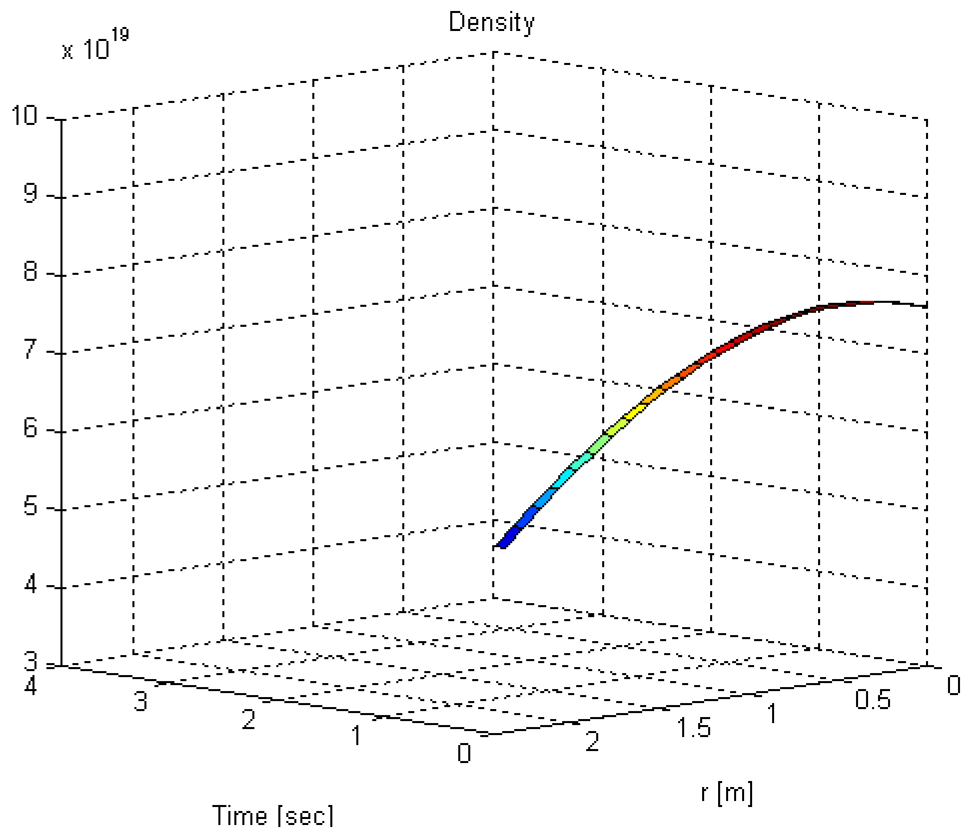
# Simulation Results

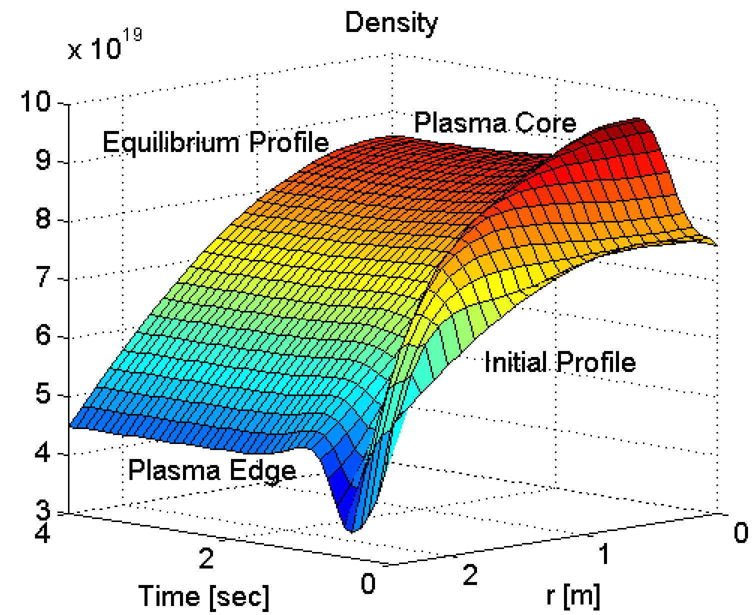



Energy Profile Evolution

# Simulation Results




Density Profile Evolution

# Simulation Results

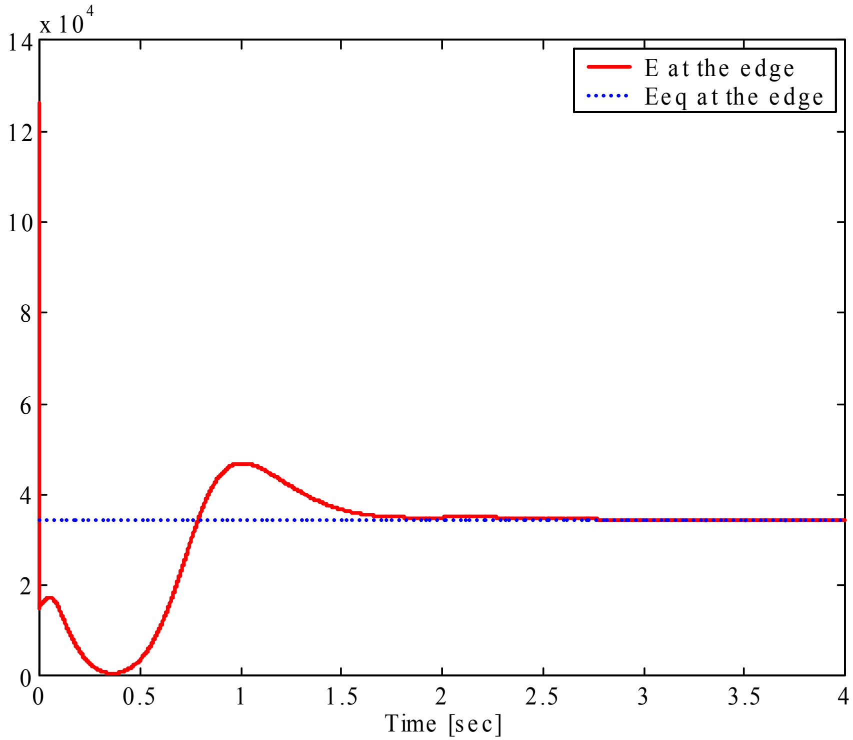



Figure 2-a: Energy Modulation at the Edge

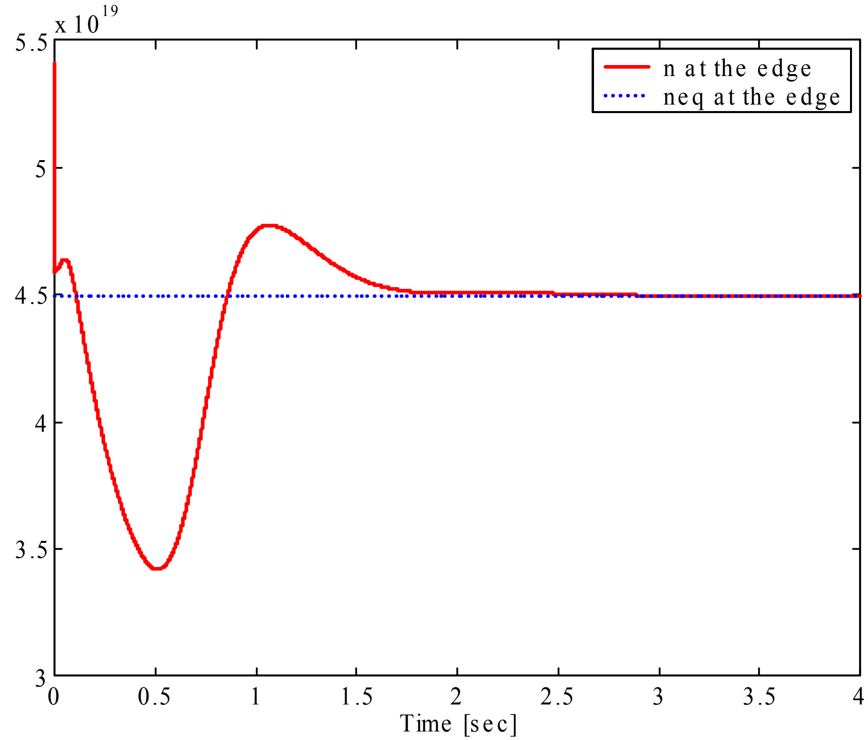



Figure 2-b: Density Modulation at the Edge

# Simulation Results

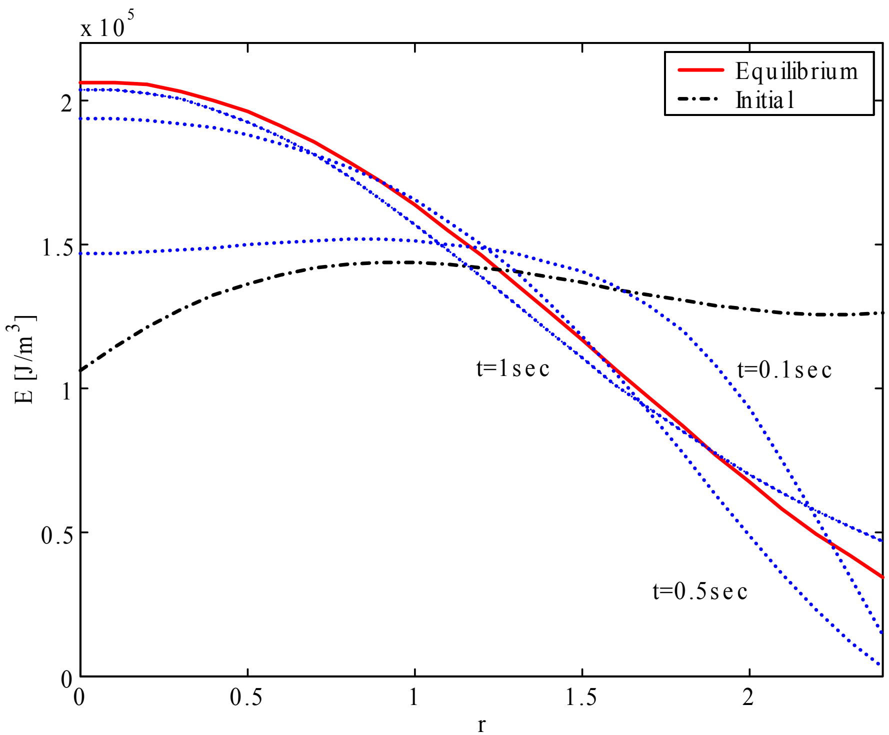



Figure 3-a: Energy Profile Evolution

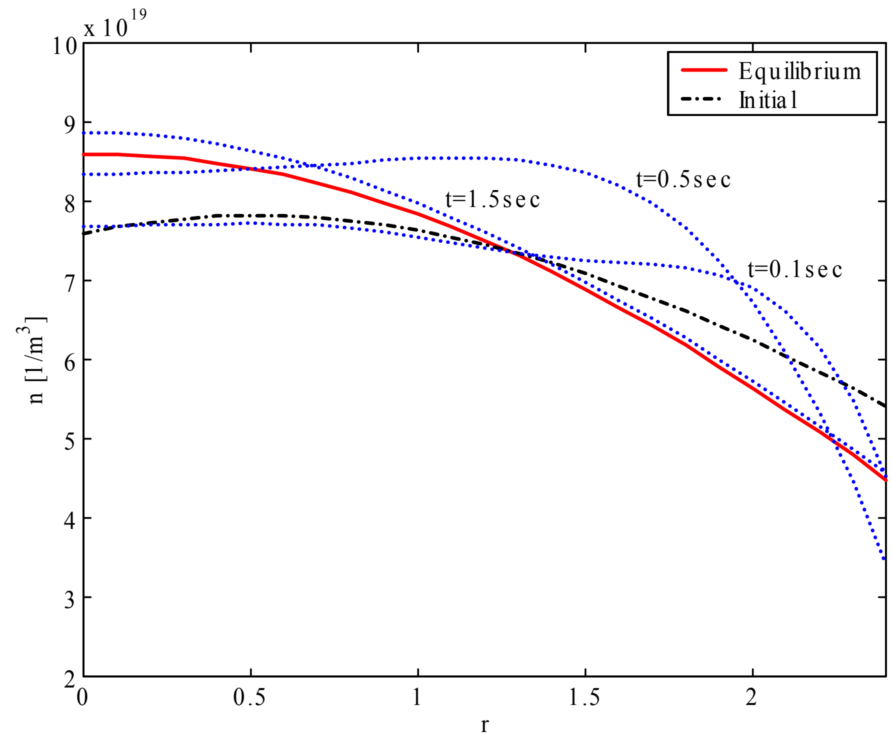
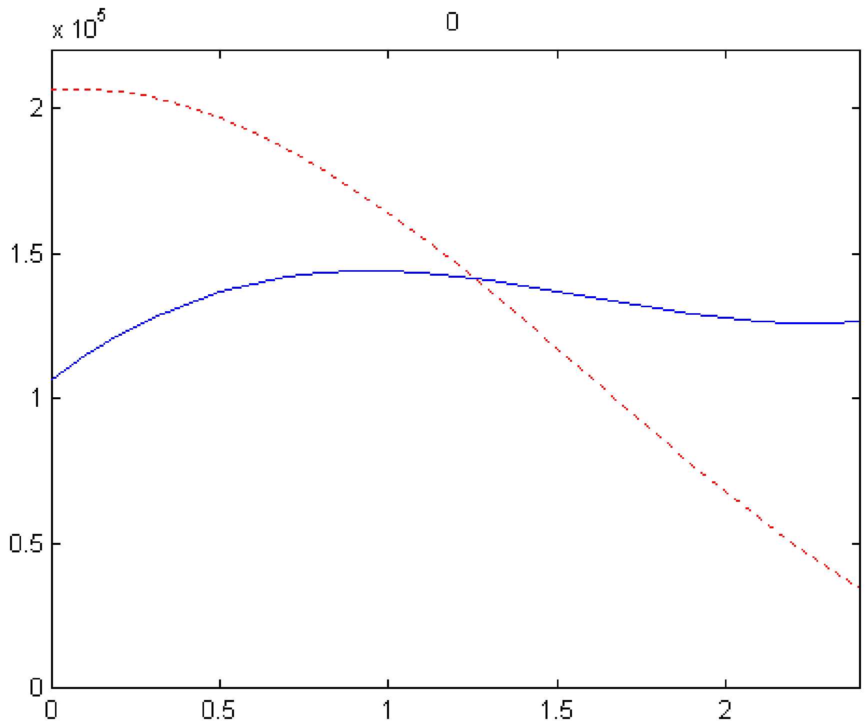
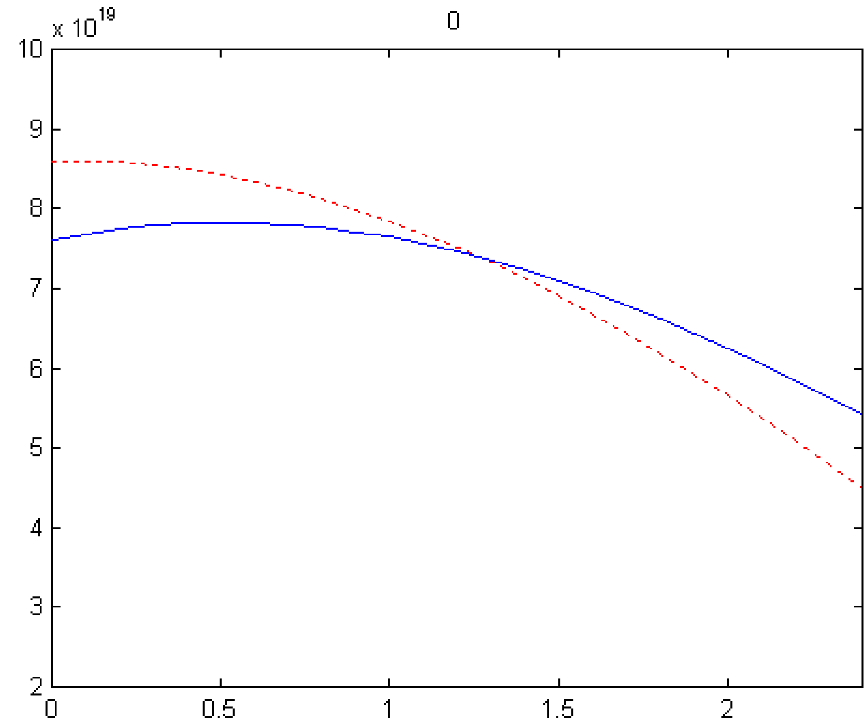



Figure 3-b: Density Profile Evolution

# Simulation Results

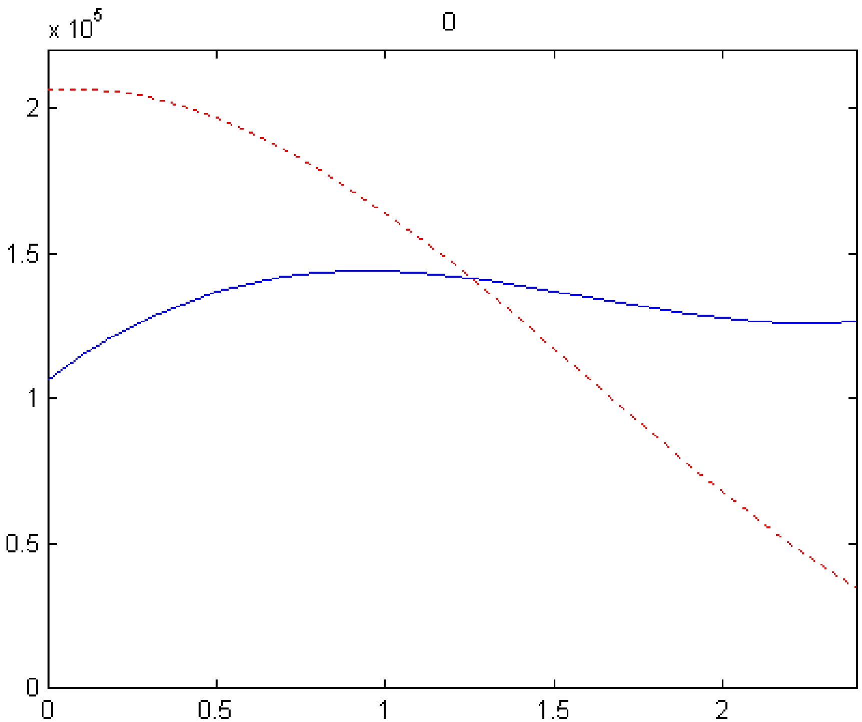



Open Loop: Energy Profile Evolution

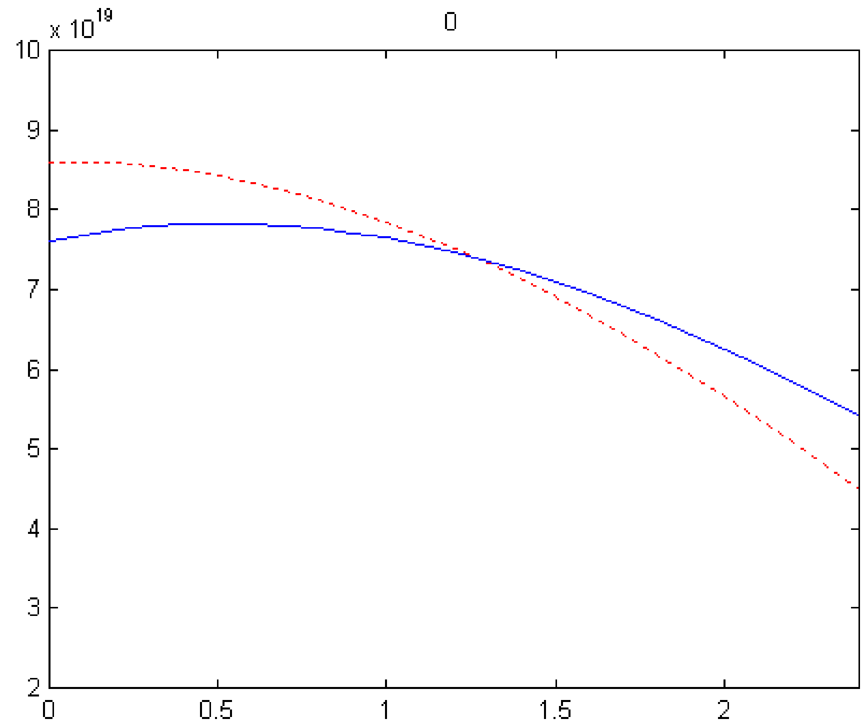



Open Loop: Density Profile Evolution

# Simulation Results

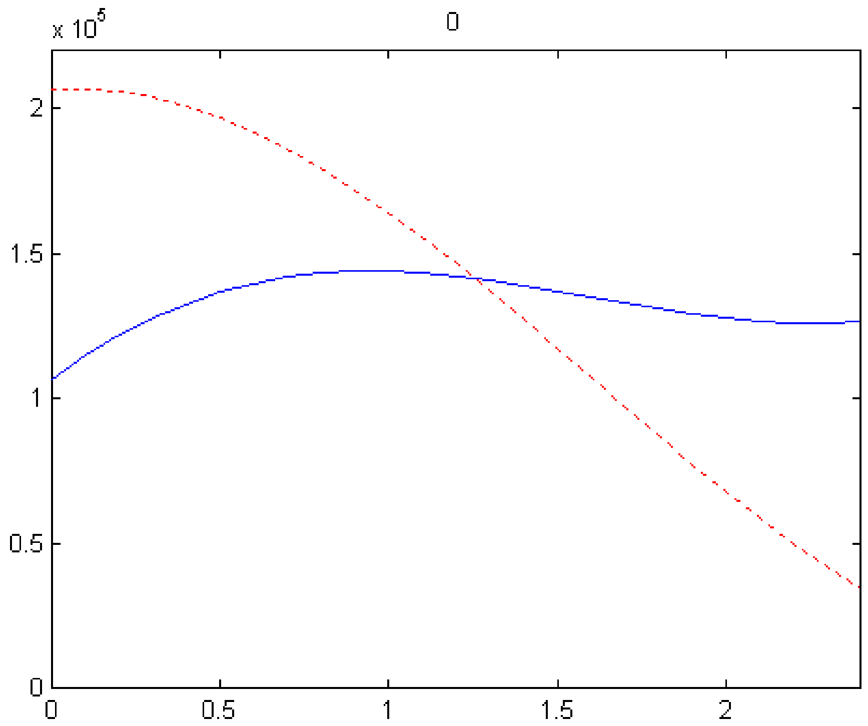



Closed Loop: Energy Profile Evolution

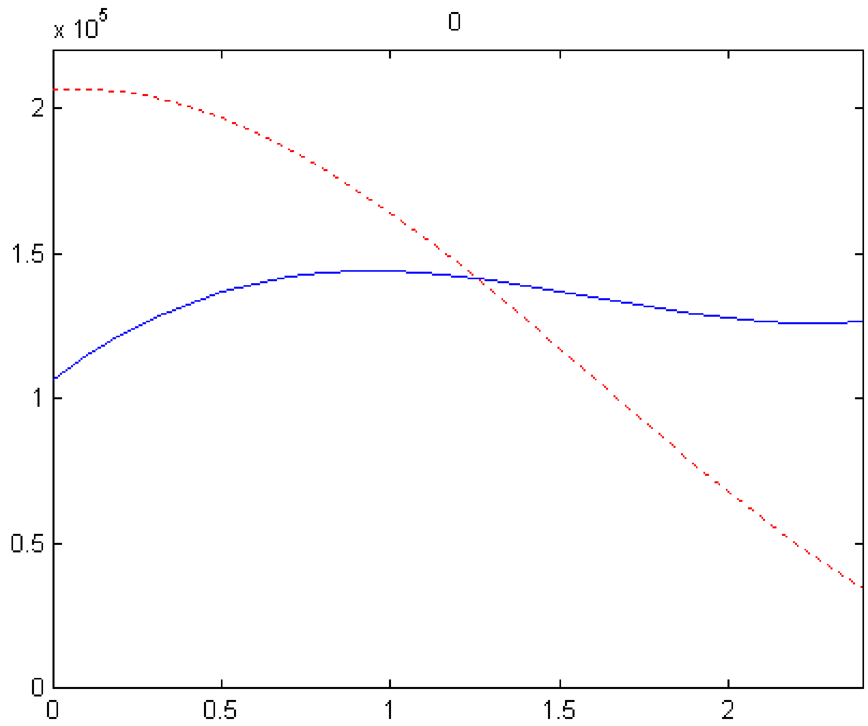



Closed Loop: Density Profile Evolution

# Simulation Results

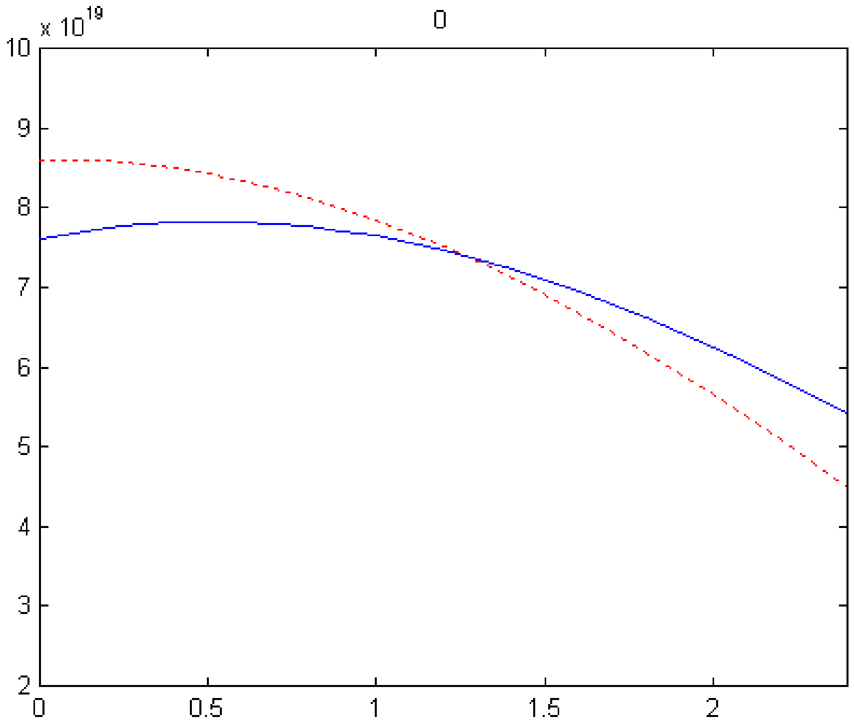



Open Loop: Energy Profile Evolution

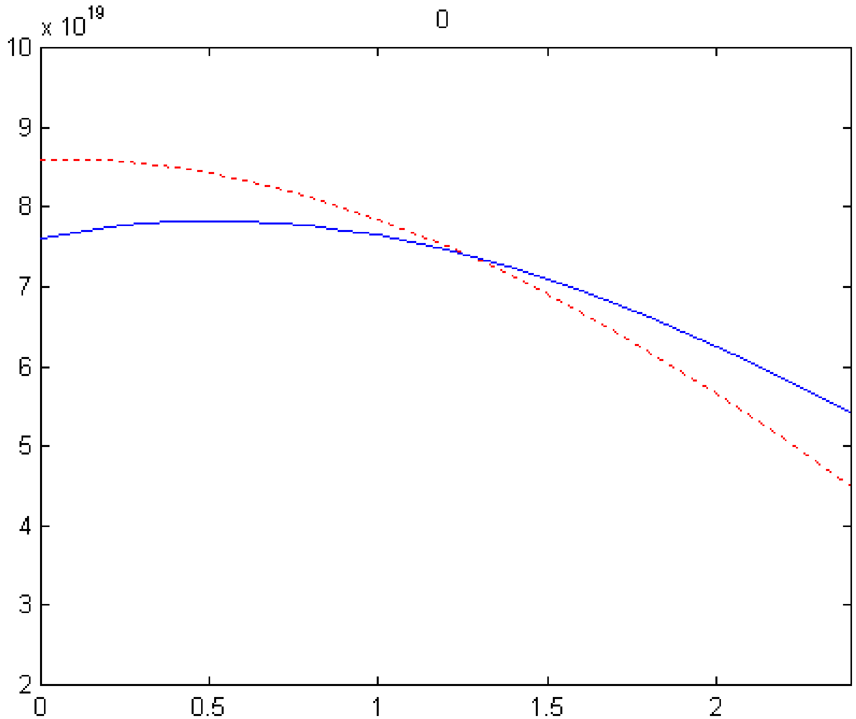



Closed Loop: Energy Profile Evolution

# Simulation Results




**Open Loop: Density Profile Evolution**




**Closed Loop: Density Profile Evolution**

# Future Work

- A zero-dimensional model of the tokamak scrape-off layer will be used as a complement of the one dimensional model for the core. This will allow to work with more realistic boundary conditions and therefore more realistic profiles. In addition, this will make possible a study of feasibility of achieving the necessary modulation of the kinetic variables at the edge by physical means.
- Actuation directly in the core of the plasma would be considered if necessary.
- A burning plasma model will be considered.
- A more updated set of correlations will be used for the physical parameters of the model.
- The problem of simultaneous control of the kinetic and current profiles will be studied.