Control of PDE Systems

Lecture 9 (Meetings 17 & 18)

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Wave Equation with "Free End" Damping

String/Cable of unit length:

$$
\begin{aligned}
u_{tt} &= u_{xx} && \text{(wave equation)} \\
u_x(0) &= 0 && \text{("free" end)} \\
u(1) &= 0 && \text{("pinned" end)}
\end{aligned}
$$

Energy/Lyapunov function

$$
E = \frac{1}{2}\left(\|u_x\|^2 + \|u_t\|^2\right)
$$

u_x	=	"shear"	potential energy
u_t	=	velocity	kinetic energy

Hyperbolic PDEs: Wave Equations

$$\begin{aligned}
\dot{E} &= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t u_{tt} dx \qquad \text{(chain rule)} \\
&= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t u_{xx} dx \\
&= \int_0^1 u_x u_{xt} dx + u_t(x) u_x(x)|_0^1 - \int_0^1 u_{tx} u_x dx \qquad \text{(integration by parts)} \\
&= u_t(x) u_x(x)|_0^1 \\
&= 0 \qquad \text{(using BCs)}
\end{aligned}$$

Conservation of energy: $E(t) \equiv E(0)$. The system is marginally/neutrally stable. Inifinitely many eigenvalues on the imaginary axis.

A classical method of asymptotically stabilizing the system is to add “boundary damping:”

$$u_x(0) = c_0 u_t(0).$$

Asymptotic stability proof by Lyapunov possible but tricky. Eigenvalue calculation easier.

First, the solution postulated as

$$u(x,t) = e^{\sigma t}\phi(x).$$

Substituting this into the PDE gives

$$\sigma^2 e^{\sigma t}\phi(x) = e^{\sigma t}\phi''(x),$$

and using the two BCs gives

$$\begin{aligned} e^{\sigma t}\phi(1) &= 0 \\ e^{\sigma t}\phi'(0) &= c_0\sigma e^{\sigma t}\phi(0). \end{aligned}$$

Sturm-Louiville problem for wave eqn with boundary damping:

$$\begin{aligned} \phi'' - \sigma^2\phi &= 0 \\ \phi'(0) &= c_0\sigma\phi(0) \\ \phi(1) &= 0. \end{aligned}$$

The solution given by

$$\phi(x) = e^{\sigma x} + Be^{-\sigma x}$$

From the BC at $x = 1$ we get

$$B = -e^{-2\sigma}.$$

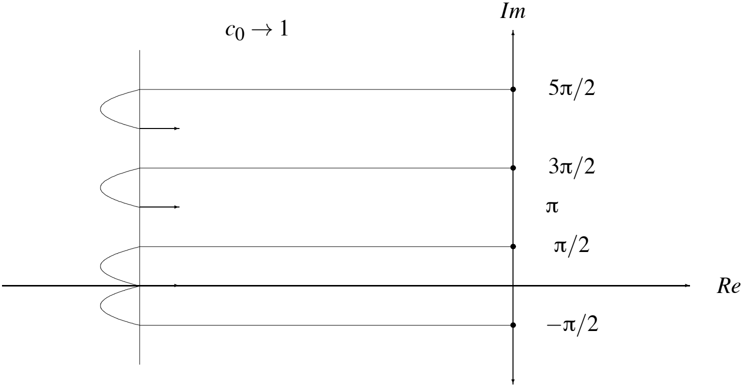
From the BC at $x = 0$ we get

$$\begin{aligned} \phi'(0) - c_0\sigma\phi(0) &= 0 \\ \sigma(1 + e^{2\sigma}) - c_0\sigma(1 - e^{2\sigma}) &= 0 \\ e^{2\sigma} &= -\frac{1 - c_0}{1 + c_0}. \end{aligned}$$

Solving for σ gives

$$\sigma = -\frac{1}{2}\ln\left|\frac{1 + c_0}{1 - c_0}\right| + j\pi \begin{cases} n + \frac{1}{2} & 0 \leq c_0 < 1 \\ n & c_0 > 1 \end{cases}$$

Eigenvalues at $-\infty$ for $c_0 = 1$ (solution $\to 0$ in finite time).



In real systems with (even the slightest) damping, the ideal c_0 is not unity. The dependence on c_0 is extremely sensitive around $c_0 = 1$.

The “boundary damper” feedback is very effective in adding damping to eigenvalues but it requires actuation on the free end $x = 0$, which is seldom feasible.

Backstepping: Actuation at the "Base"

Suppose we apply the "boundary damper" feedback using an active actuator at the base, while keeping the other end of the string/cable free:

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(0) &= 0 \\ u_x(1) &= -c_1 u_t(1)\,, \quad \text{where } c_1 > 0. \end{aligned}$$

(The sign of the gain must change to accommodate the switch from one boundary to the other, which is equivalent to the reversal of the direction of the x axis.)

Due to the Neumann BC the system has one eigenvalue at the origin $\sigma = 0$.

As a result the system has any arbitrary constant $u(x) =$ const. as an equilibrium profile.

To deal with this multitude of arbitrary equilibriums a more sophisticated (backstepping) controller is needed at $x = 1$ if the boundary condition at $x = 0$ is to remain free.

Hyperbolic PDEs: Wave Equations

Target system for the backstepping controller:

$$\begin{aligned} w_{tt} &= w_{xx} \\ w_x(0) &= c_0 w(0) \\ w_x(1) &= -c_1 w_t(1)\ , \quad \text{where } c_1 > 0. \end{aligned}$$

The BC $w_x(0) = c_0 w(0)$ doesn't use ∂_t, i.e., it is not of "damping" type but of "Robin" type.

The idea with the BC $w_x(0) = c_0 w(0)$ is to use large c_0 to make it behave like $w(0) \approx 0$.

Lyapunov function for target system:

$$V = \frac{1}{2}\left(\|w_x\|^2 + \|w_t\|^2 + c_0 w^2(0)\right) + \delta \int_0^1 (1+x) w_x(x) w_t(x) dx$$

Positive definiteness: With Poincare's and Young's inequalities, one can show that for sufficiently small δ, $\exists\, m_1, m_2 > 0$ s.t.

$$m_1 U \leqslant V \leqslant m_2 U\ , \quad \text{where} \quad U = \|w_x\|^2 + \|w_t\|^2 + w^2(0)$$

Hyperbolic PDEs: Wave Equations

$$
\begin{aligned}
\dot{V} &= \int_0^1 w_x w_{tx} dx + \int_0^1 w_t w_{tt} dx + c_0 w(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{tt}) dx
\end{aligned}
$$

substituting target system

$$
\begin{aligned}
&= \int_0^1 w_x w_{tx} dx + \int_0^1 w_t w_{xx} dx + w_x(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{xx}) dx
\end{aligned}
$$

integration by parts

$$
\begin{aligned}
&= \int_0^1 w_x w_{tx} dx + w_t w_x |_0^1 - \int_0^1 w_t w_{xt} dx + w_x(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{xx}) dx
\end{aligned}
$$

canceling terms

$$
\begin{aligned}
&= \delta \left(\int_0^1 w_{xt} w_t dx + \int_0^1 w_x w_{xx} dx + \int_0^1 x w_{xt} w_t dx + \int_0^1 x w_x w_{xx} dx \right) \\
&\quad + w_t(1) w_x(1)
\end{aligned}
$$

Hyperbolic PDEs: Wave Equations

Notice that $w_{xt}w_t dx = \frac{d}{dx}\frac{w_t^2}{2}$ and $w_x w_{xx} dx = \frac{d}{dx}\frac{w_x^2}{2}$ and use integration by parts on the latter two integrals involving an extra x term:

$$\begin{aligned}\dot{V} &= w_t(1)w_x(1) + \frac{\delta}{2}\left[(1+x)(w_x^2+w_t^2)\right]\Big|_0^1 - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right] \\ &= -c_1 w_t^2 + \delta(w_t^2(1)+w_x^2(1)) - \frac{\delta}{2}\left[w_x^2(0)+w_t^2(0)\right] - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right]\end{aligned}$$

$$\dot{V} = -\left(c_1 - \delta(1+c_1^2)\right)w_t^2(1) - \frac{\delta}{2}\left(w_t^2(0) + c_0^2 w^2(0)\right) - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right]$$

which is negative definite for $\delta < \frac{c_1}{1+c_1^2}$. One can further show that

$$U(t) \leq M e^{-t/M} U(0)$$

for some possibly large M.

This exponential stability result legitimizes our "target system."

Hyperbolic PDEs: Wave Equations

Backstepping Design. The transformation

$$w(x) = u(x) + c_0 \int_0^x u(y)dy$$

and the boundary controller

$$u_x(1) = -c_1 u_t(1) - c_0 u(1) - c_1 c_0 \int_0^1 u_t(y)dy$$

transform $u_{tt} = u_{xx},\ u_x(0) = 0$ into $w_{tt} = w_{xx},\ w_x(0) = c_0 w(0),\ w_x(1) = -c_1 w_t(1)$.

<u>Homework:</u> Prove this result.

So, $k(x,y) = c_0$!

Gain selection guideline: c_0 large and c_1 around 1.

Term-by-term discussion: $-c_1 u_t(1) - c_0 u(1)$ is PD control; $-c_1 c_0 \int_0^1 u_t(y)dy$ is a spatially averaged velocity and is a backstepping "damping" term.

Dirichlet implementation:

$$u(1) = -\frac{1}{c_1 s + c_0}[u_x(1)] - \frac{s}{c_1 s + c_0}\left[\int_0^1 u(y)dy\right]$$

First Order Hyperbolic PDEs

Traffic flow, chemical reactors, heat exchangers, delays.

The general first order hyperbolic PDE tractable by backstepping:

$$\begin{aligned} u_t &= u_x + g(x)u(0) + \int_0^x f(x,y)u(y)dy \\ u(1) &= \text{control}. \end{aligned}$$

Only one spatial derivative $\rightarrow$ only one boundary condition.

For g or f positive and large $\rightarrow$ open-loop unstable.

Transformation and boundary controller

$$\begin{aligned} w(x) &= u(x) - \int_0^x k(x,y)u(y)dy \\ u(1) &= \int_0^1 k(1,y)u(y)\,dy. \end{aligned}$$

First-Order Hyperbolic PDEs and Delay Equations

Target system

$$\begin{aligned} w_t &= w_x \\ w(1) &= 0. \end{aligned}$$

Solution

$$w(x,t) = \begin{cases} w_0(t+x) & 0 \leq t < 1 \\ 0 & t \geq 1, \end{cases}$$

where $w_0(x)$ is the initial condition. Pure delay—converges to zero in finite time.

Kernel PDE (well posed):

$$\begin{aligned} k_x + k_y &= \int_y^x k(x,\xi) f(\xi,y) d\xi - f(x,y) \\ k(x,0) &= \int_0^x k(x,y) g(y) dy - g(x)\,. \end{aligned}$$

Example 1:

$$u_t = u_x + g e^{bx} u(0), \tag{1}$$

where g and b are constants. In this case, the kernel equation becomes

$$k_x + k_y = 0, \tag{2}$$

which has a general solution $k(x, y) = \phi(x - y)$. If we plug this solution into the BC equation, we get the integral equation

$$\phi(x) = \int_0^x g e^{by} \phi(x - y) dy - g e^{bx}. \tag{3}$$

The solution to this equation can be obtained by applying Laplace Transform in x:

$$\phi(s) = -\frac{g}{s - b - g}, \tag{4}$$

and after taking the inverse Laplace transform, $\phi(x) = -g e^{(b+g)x}$. Therefore, the solution to the kernel PDE is $k(x, y) = -g e^{(b+g)(x-y)}$ and the controller given by

$$u(1) = \int_0^1 k(1, y) u(y) dy. \tag{5}$$

Example 2:

$$u_t = u_x + \int_0^x f e^{b(x-y)} u(y) dy, \tag{6}$$

where f and b are constants. In this case, the kernel equations become

$$\begin{aligned} k_x + k_y &= \int_y^x k(x,\xi) f e^{b(\xi - y)} d\xi - f e^{b(x-y)}, & (7) \\ k(x,0) &= 0. & (8) \end{aligned}$$

After we differentiate (7) with respect to y, the integral term is eliminated:

$$k_{xy} + k_{yy} = -fk - bk_x - bk_y. \tag{9}$$

Since we now increased the order of the equation, we need an extra boundary condition. We get it by setting $y = x$ in (7):

$$\frac{d}{dx} k(x,x) = k_x(x,x) + k_y(x,x) = -f, \tag{10}$$

which, after integration, becomes $k(x,x) = -fx$.

Introducing the change of variables

$$k(x,y) = p(z,y)e^{b(z-y)/2}, \qquad z = 2x - y, \tag{11}$$

we get the following PDE for $p(z,y)$:

$$p_{zz}(z,y) - p_{yy}(z,y) = fp(z,y), \tag{12}$$

$$p(z,0) = 0, \tag{13}$$

$$p(z,z) = -fz. \tag{14}$$

The solution is given by

$$p(z,y) = -2fy\frac{I_1(\sqrt{f(z^2-y^2)})}{\sqrt{f(z^2-y^2)}}, \tag{15}$$

or, in the original variables,

$$k(x,y) = -fe^{b(x-y)}y\frac{I_1(2\sqrt{fx(x-y)})}{\sqrt{fx(x-y)}}, \tag{16}$$

and the controller given by

$$u(1) = \int_0^1 k(1,y)u(y)dy. \tag{17}$$

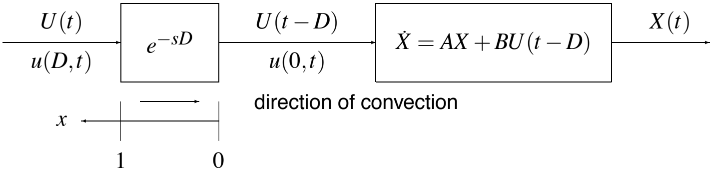
Systems with Delay

$$\dot{X} = AX + BU(t-D)$$

Assume: (A,B) controllable and matrix K found such that $A+BK$ is Hurwitz.

A hyperbolic PDE representation:

$$\begin{aligned} \dot{X} &= AX + Bu(0,t) \\ u_t &= u_x \\ u(D,t) &= U(t) \end{aligned}$$



Note that $u(x,t) = U(t+x-D)$.

Consider the backstepping transformation

$$w(x) = u(x) - \int_0^x q(x,y)u(y)dy - \gamma(x)^T X$$

and the target system

$$\begin{aligned} \dot{X} &= (A+BK)X + Bw(0) \\ w_t &= w_x \\ w(D) &= 0\,. \end{aligned}$$

Since w becomes zero in finite time, the w-system is exponentially stable.

As usual, let us calculate the time and spatial derivatives of the transformation:

$$\begin{aligned} w_x &= u_x - q(x,x)u(x) - \int_0^x q_x(x,y)u(y)dy - \gamma'(x)^T X \\ w_t &= u_t - \int_0^x q(x,y)u_t(y)dy - \gamma(x)^T \left[AX + Bu(0)\right] \end{aligned}$$

We get three conditions:

$$\begin{aligned} q_x + q_y &= 0 \\ q(x,0) &= \gamma(x)^T B \\ \gamma' &= A^T \gamma \end{aligned}$$

The first two conditions form a familiar first order hyperbolic PDE and the third one is a simple ODE.

To find the initial condition for the ODE, we set $x = 0$ in $w(x)$, which gives $w(0) = u(0) - \gamma(0)^T X$, and hence

$$\dot{X} = AX + Bu(0) + B\left(K - \gamma(0)^T\right)X.$$

We thus get $\gamma(0) = K^T$.

Therefore the ODE is

$$\begin{aligned} \gamma' &= A^T\gamma \\ \gamma(0) &= K^T \end{aligned}$$

The solution is

$$\gamma(x)^T = K\mathrm{e}^{Ax}$$

The q-PDE is

$$\begin{aligned} q_x + q_y &= 0 \\ q(x,0) &= \gamma(x)^T B \end{aligned}$$

The solution is given explicitly:

$$q(x,y) = K\mathrm{e}^{A(x-y)}B$$

First-Order Hyperbolic PDEs and Delay Equations

This gives the control law:

$$u(D) = \int_0^D K e^{A(D-y)} B u(y) dy + K e^{AD} X$$

or

$$U(t) = K \left[e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} B U(\theta) d\theta \right]$$

Same controller as in Artstein (1982) but a better proof (complete Lyapunov function).

The equivalent of Smith Predictor for unstable systems.