Control of PDE Systems

Lecture 8 (Meetings 15 & 16)

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Motion Planning for PDEs

- Combined control solution: Feedforward (FF) + Feedback (FB)
- Trajectory generation, or motion planning $\rightarrow$ open-loop control for PDEs
- Reference trajectory + Combined FF+FB $\rightarrow$ Trajectory stabilization
- Trajectory tracking: state reference (relevant) $\rightarrow$ output reference (important)
  - Start from output reference trajectory, e.g. $u(0,t) = u^r(0,t)$
  - Generates the state trajectory $u^r(x,t)$ for all x
    - Including $x = 1 \rightarrow$ control reference $u^r(1,t)$
  - Combine this FF results with FB control law
    - Stabilize the trajectory $u^r(x,t)$
    - Force $u(0,t) = u^r(0,t)$

Trajectory Generation

We present the ideas through several examples.

Consider the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u_x(0) &= 0 \\ u(0) &= \text{system output} \\ u(1) &= \text{system input} \end{aligned}$$

The objective is to follow the **output** trajectory

$$u^r(0,t) = 2-(t-1)^2 = 1+2t-t^2$$

The first step is to construct the full state trajectory $u^r(x,t)$ which satisfies the PDE. Then add a feedback law that stabilizes that solution.

Motion Planning for PDEs

We postulate the state trajectory in the form:

$$u^r(x,t) = \sum_{k=0}^{\infty} a_k(t) \frac{x^k}{k!}.$$

This is a Taylor series in x with time varying coefficients $a_k(t)$ that will be determined from the PDE, boundary condition, and desired trajectory.

From the desired trajectory we get

$$u^r(0,t) = a_0(t) = 1 + 2t - t^2.$$

The boundary condition gives

$$u_x^r(0,t) = a_1(t) = 0.$$

Substituting $u^r(x,t)$ into the PDE we get:

$$\begin{aligned}
\sum_{k=0}^{\infty} \dot{a}_k(t)\frac{x^k}{k!} &= \frac{\partial^2}{\partial x^2}\sum_{k=0}^{\infty} a_k(t)\frac{x^k}{k!} \\
&= \sum_{k=2}^{\infty} a_k(t)\frac{k(k-1)x^{k-2}}{k!} \\
&= \sum_{k=2}^{\infty} a_k(t)\frac{x^{k-2}}{(k-2)!} \\
&= \sum_{k=0}^{\infty} a_{k+2}(t)\frac{x^k}{k!}.
\end{aligned}$$

We get the recursive relationship

$$a_{k+2}(t) = \dot{a}_k(t)$$

The recursion yields

$$\begin{array}{rclll} a_0 &=& 1+2t-t^2, & a_1 = 0 & \\ a_2 &=& 2-2t, & a_3 = 0 & \\ a_4 &=& -2, & a_5 = 0 & \\ a_6 &=& 0, & a_i = 0 & \text{for } i > 6. \end{array}$$

This gives the reference trajectory

$$u^r(x,t) = 1 + 2t + t^2 + (1-t)x^2 - \frac{1}{12}x^4,$$

and the input signal

$$u^r(1,t) = \frac{23}{12} + t - t^2.$$

Remark. Perfect trajectory obtained only if the initial condition of the plant agrees with the initial condition of the trajectory, that is, $u(x,0) = 1 + x^2 - \frac{1}{12}x^4$.

Example 2. Reaction-diffusion equation

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Desired output reference signal

$$u^r(0,t) = \mathrm{e}^{\alpha t}.$$

From the boundary condition we have

$$a_1(t) = 0,$$

and from the PDE

$$a_{k+2}(t) = \dot{a}_k(t) + \lambda a_k(t).$$

These conditions give

$$\begin{aligned} a_{2k+1} &= 0 \\ a_{2k+2} &= \dot{a}_{2k} - \lambda a_{2k} \\ a_2 &= (\alpha - \lambda)\mathrm{e}^{\alpha t} \\ a_4 &= (\alpha - \lambda)^2 \mathrm{e}^{\alpha t} \\ a_{2k} &= (\alpha - \lambda)^k \mathrm{e}^{\alpha t} \end{aligned}$$

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The state trajectory is

$$
\begin{aligned}
u^r(x,t) &= \sum_{k=0}^{\infty} (\alpha-\lambda)^k \mathrm{e}^{\alpha t} \frac{x^{2k}}{2k!} \\
&= \mathrm{e}^{\alpha t} \sum_{k=0}^{\infty} \frac{(\sqrt{\alpha-\lambda}x)^{2k}}{2k!} \\
&= \mathrm{e}^{\alpha t} \begin{cases} \cosh(\sqrt{\alpha-\lambda}x) & \alpha \geq \lambda \\ \cos(\sqrt{\alpha-\lambda}x) & \alpha < \lambda \end{cases}
\end{aligned}
$$

The reference input is

$$
u^r(1,t) = \mathrm{e}^{\alpha t} \cosh(\sqrt{\alpha-\lambda}).
$$

Useful formulae when calculating trajectories for sinusoidal outputs:

$$
\begin{aligned}
\cosh(a) &= \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!}, \qquad \sinh(a) = \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!} \\
\cosh(ja) &= \cos(a), \qquad \sinh(ja) = j\sin(a), \qquad \sin(ja) = j\sinh(a).
\end{aligned}
$$

Example 8. First order hyperbolic PDE

$$\begin{aligned} u_t &= u_x + g u(0) \\ u^r(1,t) &= \text{control} \\ u^r(0,t) &= \sin(\omega t) = \operatorname{Im}\left\{ \mathrm{e}^{j\omega t} \right\} = a_0(t) \end{aligned}$$

Since this is a first order PDE, the only boundary condition is the one that is available for control.

From the PDE we get

$$\begin{aligned} a_1 &= \dot{a}_0 - g a_0 \\ &= \operatorname{Im}\left\{ (j\omega - g)\mathrm{e}^{j\omega t} \right\} \end{aligned}$$

$$\begin{aligned} a_{i+1} &= \dot{a}_i \\ a_k(t) &= \operatorname{Im}\left\{ (j\omega - g)(j\omega)^{k-1} \mathrm{e}^{j\omega t} \right\} \\ &= \operatorname{Im}\left\{ \left(1 - \frac{g}{j\omega}\right) (j\omega)^k \mathrm{e}^{j\omega t} \right\} \end{aligned}$$

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State reference

$$
\begin{aligned}
u^r(x,t) &= \mathrm{Im}\left\{\left[1+\left(1-\frac{g}{j\omega}\right)\sum_{k=1}^{\infty}\frac{(j\omega x)^k}{k!}\right]\mathrm{e}^{j\omega t}\right\} \\
& \quad \text{add and subtract } \frac{g}{j\omega} \\
&= \mathrm{Im}\left\{\left[\frac{g}{j\omega}+\left(1-\frac{g}{j\omega}\right)\mathrm{e}^{j\omega x}\right]\mathrm{e}^{j\omega t}\right\} \\
&= \mathrm{Im}\left\{\frac{g}{j\omega}\mathrm{e}^{j\omega t}+\frac{j\omega-g}{j\omega}\mathrm{e}^{j\omega(t+x)}\right\}
\end{aligned}
$$

$$
u^r(x,t) = -\frac{g}{\omega}[\cos(\omega t)-\cos(\omega(t+x))]+\sin(\omega(t+x))
$$

Input reference

$$
u^r(1,t) = \frac{g}{\omega}[\cos(\omega(t+1))-\cos(\omega t)]+\sin(\omega(t+1)).
$$

Trajectory Tracking

Trajectory Tracking = stabilization of reference trajectory with feedback control.

Consider the previous example

$$\begin{aligned} \text{plant:} \quad u_t &= u_x + g u(0) \\ \text{trajectory:} \quad u^r(x,t) &= \frac{g}{\omega}[\cos(\omega(t+x)) - \cos(\omega t)] + \sin(\omega(t+x)) \end{aligned}$$

Stabilizing controller

$$u(1,t) - u^r(1,t) = \int_0^1 k(1,y)[u(y,t) - u^r(y,t)]dy$$

$$\begin{aligned} \text{transformation:} \quad w(x,t) &= u(x,t) - u^r(x,t) - \textstyle\int_0^x k(x,y)[u(y,t) - u^r(y,t)]dy \\ \text{kernel:} \quad k(x,y) &= -g e^{g(x-y)} \\ \text{target system:} \quad w_t &= w_x \qquad \text{(unit delay with zero input)} \\ w(1) &= 0 \end{aligned}$$

Motion Planning for PDEs

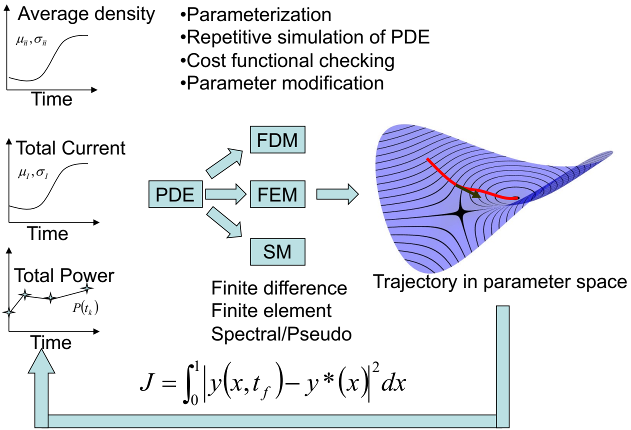
Stabilizing controller

$$
\begin{aligned}
u(1,t) - u^r(1,t) &= \int_0^1 k(1,y)[u(y,t) - u^r(y,t)]dy \\
u(1,t) &= u^r(1,t) + \int_0^1 k(1,y)[u(y,t) - u^r(y,t)]dy \\
&= \underbrace{u^r(1,t) - \int_0^1 k(1,y)u^r(y,t)dy}_{\text{feedforward (fcn of } t\text{)}} + \underbrace{\int_0^1 k(1,y)u(y,t)dy}_{\text{feedback}}
\end{aligned}
$$

$$
\begin{aligned}
\text{Ffwd} &= \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] + \sin(\omega(t+1)) \\
&\quad + \int_0^1 g e^{g(1-y)} \left\{ \frac{g}{\omega}[\cos(\omega(t+y)) - \cos(\omega t)] + \sin(\omega(t+x)) \right\} dy \\
&= \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] + \sin(\omega(t+1)) - \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] \\
&= \sin(\omega(t+1))
\end{aligned}
$$

which is $u^r(0,t) = \sin(\omega t)$ advanced by one time unit!

Thus, it suffices to determine the reference trajectory for the target system (rather than for the complicated original system). This is true in general.



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Feedforward (FF) Control Optimization Problem Defined as Tradeoff Between Multiple Objectives

- Control Synthesis: feedforward controller, feedback controllers, state observers.

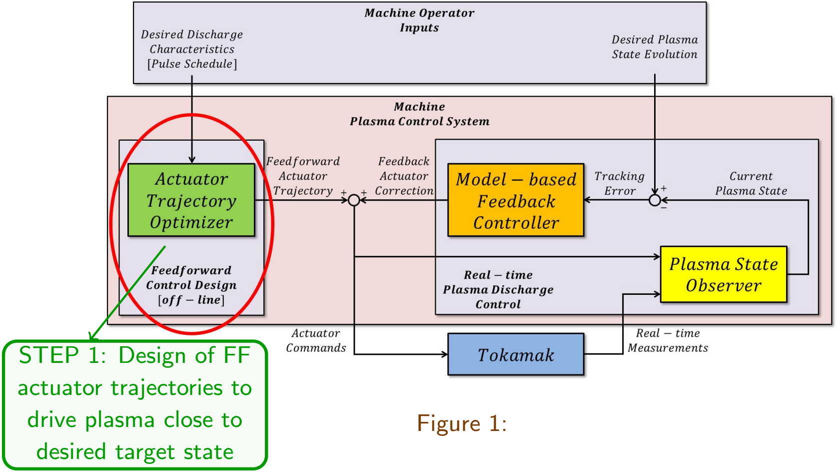


Figure 1:

Feedforward (FF) Control Optimization Problem Defined as Tradeoff Between Multiple Objectives

- **Objective:** Reach **target plasma state** at some time t_{targ} by designing **actuator waveforms** subject to **plasma dynamics** and **constraints**.
  - Defines nonlinear, constrained optimization problem.
- **Target state:** Defined in terms of q profile and β_N.
  - Proximity of achieved state to target formulated into cost functional (J).
  - To-be-minimized cost function is weighted sum of three objectives

$$J = k_q J_q(q(t_{\text{targ}})) + k_{ss} J_{ss}(\dot{\psi}(t_{\text{targ}})) + k_{\beta_N} J_{\beta_N}(\beta_N(t_{\text{targ}}))$$

  - Target q-profile objective

$$J_q(q(t_{\text{targ}})) = \int_0^1 W_q(\hat{\rho}) \left[q_{\text{targ}}(\hat{\rho}) - q(\hat{\rho}, t_{\text{targ}})\right]^2 d\hat{\rho}$$

  - Stationarity objective

$$J_{ss}(\dot{\psi}(t_{\text{targ}})) = \int_0^1 W_{ss}(\hat{\rho}) \left[g_{ss}(\hat{\rho}, t_{\text{targ}})\right]^2 d\hat{\rho}$$

$$g_{ss}(\hat{\rho}, t) = -2\pi \frac{\partial U_p}{\partial \hat{\rho}} \qquad U_p = \frac{\partial \psi}{\partial t} = \text{loop voltage}$$

  - Target β_N objective

$$J_{\beta_N}(\beta_N(t_{\text{targ}})) = \left[\beta_{N_{\text{targ}}} - \beta_N(t_{\text{targ}})\right]^2$$

Solution of Feedforward (FF) Control Optimization Problem Requires Minimization under Constraints

- An open-loop (feedforward) control policy u_{FF} is obtained via nonlinear optimization subject to constraints. Returns target trajectory ψ_{FF} (q_{FF}).

$$\min_{\boldsymbol{\alpha}} \quad J(\psi_{FF}, \dot{\psi}_{FF}, \beta_N) \quad \Big\} \text{ Cost Function (Optimization Objective)}$$

$$\text{s.t.} \quad \dot{\psi}_{FF} = f_\psi(\psi_{FF}, u_{\text{FF}}), \psi_{FF}(t_0) = \psi_0 \quad \Big\} \text{MDE}$$

$$\dot{W}_{FF} = f_W(W_{FF}, u_{\text{FF}}) \quad \Big\} \text{ Energy Balance}$$

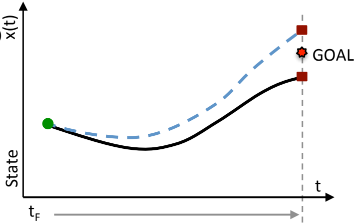
$$\beta_N(t) \le \beta_{N_{\text{max}}}, q \ge 1 \quad \Big\} \text{ State Constraint, MHD Stability Limit}$$

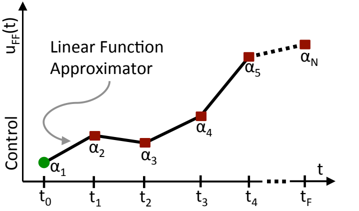
$$\bar{n}_e|_{20} \le I_{\text{p}}/\pi a^2 \quad \Big\} \text{ State Constraint, Density Limit}$$

$$P_{\text{tot}}(t) \ge P_{\text{tot}_{\text{min}}} \quad \Big\} \text{ State Constraint, Prevent H} \rightarrow \text{L Transition}$$

$$u_{\text{FF}}(t) \in \mathcal{U} \quad \Big\} \text{ Input Constraint, Saturation / Rate Limit}$$

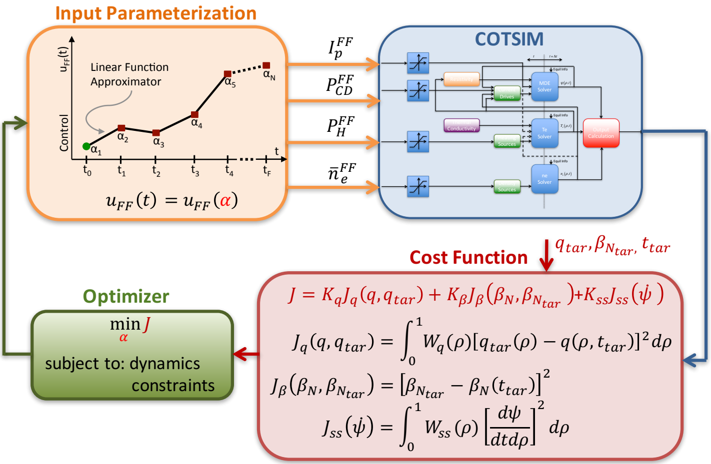
$$u_{\text{FF}}(t) = u_{\text{FF}}(\boldsymbol{\alpha}) \quad \Big\} \text{ Linear Function Approximator}$$





Solution of Feedforward (FF) Control Optimization Problem Requires Minimization under Constraints

- **COTSIM enables systematic model-based scenario planning based on nonlinear constrained optimization with arbitrary cost function.**



- **FF Design (offline): Arbitrary model complexity (COTSIM$\rightarrow$TRANSP)**