Control of PDE Systems

Lecture 7 (Meetings 13 & 14)

Eugenio Schuster

schuster@lehigh.edu
Mechanical Engineering and Mechanics
Lehigh University

Material provided by Prof. Miroslav Krstic and Dr. Andrey Smyshlyaev (UCSD)

Observers

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Possible input-output architectures:

- Anti-collocated: $u(0)$ measured and $u(1)$ or $u_x(1)$ actuated
- Collocated: $u_x(1)$ measured and $u(1)$ actuated (fluid problems)

  $u(1)$ measured and $u_x(1)$ actuated (thermal problems)

Anti-Collocated Setup

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Input: $u(1)$ Output: $u(0)$

Observer

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + p_1(x)[u(0) - \hat{u}(0)] \\ \hat{u}_x(0) &= p_{10}[u(0) - \hat{u}(0)] \\ \hat{u}(1) &= u(1) \end{aligned}$$

The function $p_1(x)$ and the constant p_{10} are observer gains.

Compare with finite dimension:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The error $\tilde{u} = u - \hat{u}$ satisfies

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + \lambda \tilde{u} - p_1(x)\tilde{u}(0) \\ \tilde{u}_x(0) &= -p_{10}\tilde{u}(0) \\ \tilde{u}(1) &= 0 \end{aligned}$$

We use the transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)\,dy$$

to convert the error system into the stable target system:

$$\begin{aligned} \tilde{w}_t &= \tilde{w}_{xx} \\ \tilde{w}_x(0) &= 0 \\ \tilde{w}(1) &= 0 \end{aligned}$$

Taking a derivative with respect to time we get

$$\begin{aligned}
\tilde{u}_t &= \tilde{w}_t - \int_0^x p(x,y)\tilde{w}_{yy}(y)\,dy \\
&= \tilde{w}_{xx} - p(x,x)\tilde{w}_x(x) + p(x,0)\tilde{w}_x(0) + \int_0^x p_y(x,y)\tilde{w}_y(y)\,dy \\
&= \tilde{w}_{xx} - p(x,x)\tilde{w}_x(x) + p_y(x,x)\tilde{w}(x) - p_y(x,0)\tilde{w}(0) - \int_0^x p_{yy}(x,y)\tilde{w}(y)\,dy
\end{aligned}$$

Taking derivatives with respect to x we get

$$\begin{aligned}
\tilde{u}_x &= \tilde{w}_x - p(x,x)\tilde{w}(x) - \int_0^x p_x(x,y)\tilde{w}(y)dy \\
\tilde{u}_{xx} &= \tilde{w}_{xx} - \frac{d}{dx}(p(x,x))\tilde{w}(x) - p(x,x)\tilde{w}_x(x) - p_x(x,x)\tilde{w}(x) - \int_0^x p_{xx}(x,y)\tilde{w}(y)dy
\end{aligned}$$

From the error system we get

$$\begin{aligned}
\tilde{u}_t - \tilde{u}_{xx} - \lambda\tilde{u} + p_1(x)\tilde{u}(0) &= 0 \\
&= [-p_y(x,0) + p_1(x)]\tilde{w}(0) + \left[2\frac{d}{dx}(p(x,x)) - \lambda\right]\tilde{w}(x) \\
&\quad + \int_0^x [p_{xx}(x,y) - p_{yy}(x,y) + \lambda p(x,y)]\tilde{w}(y)dy
\end{aligned}$$

This gives 3 conditions:

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) \\ \frac{d}{dx} p(x,x) &= \frac{\lambda}{2} \\ p_1(x) &= p_y(x,0) \end{aligned}$$

Boundary conditions of the error system give 2 more conditions

$$\begin{aligned} \tilde{u}_x(0) &= -p_{10}\tilde{u}(0) && \Rightarrow \quad p(0,0) = p_{10} \\ \tilde{u}(1) &= 0 && \Rightarrow \quad p(1,y) = 0 \end{aligned}$$

Observer kernel PDE

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) \\ p(1,y) &= 0 \\ p(x,x) &= -\frac{\lambda}{2}(1-x) \end{aligned}$$

Observer gains

$$\begin{aligned} p_1(x) &= p_y(x,0) \\ p_{10} &= p(0,0) \end{aligned}$$

Note: How do we obtain conditions from boundary conditions?

$$\begin{aligned}\tilde{u}(x) &= \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)dy && (1)\\ \tilde{u}_x(x) &= \tilde{w}_x(x) - p(x,x)\tilde{w}(x) - \int_0^x p_x(x,y)\tilde{w}(y)dy && (2)\end{aligned}$$

For $x = 0$, we have

$$\begin{aligned}\tilde{u}(0) &= \tilde{w}(0) && (3)\\ \tilde{u}_x(0) &= \tilde{w}_x(0) - p(0,0)\tilde{w}(0) = -p_{10}\tilde{u}(0) && (4)\end{aligned}$$

Since $\tilde{w}_x(0) = 0$ and $\tilde{u}(0) = \tilde{w}(0)$, we finally have

$$p(0,0) = p_{10} \tag{5}$$

For $x = 1$, we have

$$\tilde{u}(1) = \tilde{w}(1) - \int_0^1 p(1,y)\tilde{w}(y)dy \tag{6}$$

Since $\tilde{u}(1) = 0$ and $\tilde{w}(1) = 0$, we finally need

$$p(1,y) = 0 \tag{7}$$

Note: How do we obtain modified boundary condition?

$$\frac{d}{dx}p(x,x) = \frac{\lambda}{2} \Rightarrow p(x,x) = \frac{\lambda}{2}x + C \tag{8}$$

Note that $p(1,y) = 0$, in particular when $y = 1$, i.e.

$$p(1,1) = 0 \tag{9}$$

Therefore,

$$p(1,1) = \frac{\lambda}{2} + C = 0 \Rightarrow C = -\frac{\lambda}{2} \tag{10}$$

and

$$p(x,x) = \frac{\lambda}{2}(x-1) \tag{11}$$

Change of variables

$$\bar{x} = 1 - y \qquad \bar{y} = 1 - x \qquad \bar{p}(\bar{x},\bar{y}) = p(x,y)$$

Observer PDE in new variables

$$\begin{aligned} \bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \lambda \bar{p}(\bar{x},\bar{y}) \\ \bar{p}(\bar{x},0) &= 0 \\ \bar{p}(\bar{x},\bar{x}) &= -\frac{\lambda}{2}\bar{x} \end{aligned}$$

The solution is

$$\bar{p}(\bar{x},\bar{y}) = -\lambda \bar{y} \frac{I_1\left(\sqrt{\lambda(\bar{x}^2-\bar{y}^2)}\right)}{\sqrt{\lambda(\bar{x}^2-\bar{y}^2)}} = -\lambda(1-x)\frac{I_1\left(\sqrt{\lambda(x-y)(2-x-y)}\right)}{\sqrt{\lambda(x-y)(2-x-y)}}$$

Observer gains

$$\begin{aligned} p_1(x) &= p_y(x,0) = -\frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) \\ p_{10} &= p(0,0) = -\frac{\lambda}{2} \end{aligned}$$

Note: How do we obtain PDE in new variables?

$$p_x = \bar{p}_{\bar{x}} \frac{d\bar{x}}{dx} + \bar{p}_{\bar{y}} \frac{d\bar{y}}{dx} = -\bar{p}_{\bar{y}} \quad (12)$$

$$p_{xx} = -\bar{p}_{\bar{y}\bar{x}} \frac{d\bar{x}}{dx} - \bar{p}_{\bar{y}\bar{y}} \frac{d\bar{y}}{dx} = \bar{p}_{\bar{y}\bar{y}} \quad (13)$$

$$p_y = \bar{p}_{\bar{x}} \frac{d\bar{x}}{dy} + \bar{p}_{\bar{y}} \frac{d\bar{y}}{dy} = -\bar{p}_{\bar{x}} \quad (14)$$

$$p_{yy} = -\bar{p}_{\bar{x}\bar{x}} \frac{d\bar{x}}{dy} - \bar{p}_{\bar{x}\bar{y}} \frac{d\bar{y}}{dy} = \bar{p}_{\bar{x}\bar{x}} \quad (15)$$

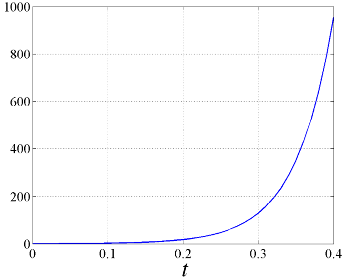
Then

$$p_{xx} - p_{yy} = -\lambda p \iff \bar{p}_{\bar{y}\bar{y}} - \bar{p}_{\bar{x}\bar{x}} = -\lambda \bar{p} \iff \bar{p}_{\bar{x}\bar{x}} - \bar{p}_{\bar{y}\bar{y}} = \lambda \bar{p} \quad (16)$$

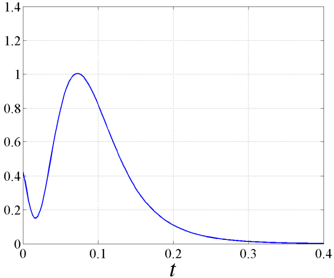
Note that $x = 1 \iff \bar{y} = 0$. Then, $p(1, y) = 0 \iff \bar{p}(\bar{x}, 0) = 0$.

Note that $x = y \iff \bar{x} = \bar{y}$. Then $\bar{p}(\bar{x}, \bar{x}) = p(x, x) = \frac{\lambda}{2}(x - 1) = -\frac{\lambda}{2}\bar{y} = -\frac{\lambda}{2}\bar{x}$.

Observer Simulation



L_2 norm of the open loop state



L_2 norm of the observer error

Collocated Setup

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Input: $u(1)$ Output: $u_x(1)$

Observer

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + p_1(x)[u_x(1) - \hat{u}_x(1)] \\ \hat{u}_x(0) &= 0 \\ \hat{u}(1) &= u(1) + p_{10}[u_x(1) - \hat{u}_x(1)] \end{aligned}$$

The error $\tilde{u} = u - \hat{u}$ satisfies

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + \lambda \tilde{u} - p_1(x)\tilde{u}_x(1) \\ \tilde{u}_x(0) &= 0 \\ \tilde{u}(1) &= -p_{10}\tilde{u}_x(1) \end{aligned}$$

Backstepping Observer for Parabolic PDEs: Design $\rightarrow$ Discretize

Using the transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_x^1 p(x,y)\tilde{w}(y)\,dy$$

we transform the error system into

$$\begin{aligned} \tilde{w}_t &= \tilde{w}_{xx} \\ \tilde{w}_x(0) &= 0 \\ \tilde{w}(1) &= 0 \end{aligned}$$

Observer gain PDE

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) \\ p(x,x) &= -\frac{\lambda}{2}x \\ p_x(0,y) &= 0 \end{aligned}$$

The observer gains are

$$\begin{aligned} p_1(x) &= p(x,1) \\ p_{10} &= 0 \end{aligned}$$

Using the change of variables

$$\bar{x} = y \qquad \bar{y} = x$$

we get the same PDE as for the control kernel

$$\begin{aligned}
\bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \lambda p(\bar{x},\bar{y}) \\
\bar{p}_{\bar{y}}(\bar{x},0) &= 0 \\
\bar{p}(\bar{x},\bar{x}) &= -\frac{\lambda}{2}\bar{x}
\end{aligned}$$

The solution is

$$p(\bar{x},\bar{y}) = -\lambda\bar{x}\frac{I_1\left(\sqrt{\lambda(\bar{x}^2-\bar{y}^2)}\right)}{\sqrt{\lambda(\bar{x}^2-\bar{y}^2)}} = -\lambda y\frac{I_1\left(\sqrt{\lambda(y^2-x^2)}\right)}{\sqrt{\lambda(y^2-x^2)}}$$

Observer gain

$$p_1(x) = -\lambda\frac{I_1\left(\sqrt{\lambda(1-x^2)}\right)}{\sqrt{\lambda(1-x^2)}} = k(1,x) \quad \text{(duality)}$$

Output Feedback

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(0) &= 0 \end{aligned}$$

Input: $u_x(1)$ Output: $u(1)$

Observer

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + \frac{\lambda x}{1-x^2} I_2\left(\sqrt{\lambda(1-x^2)}\right)[u(1) - \hat{u}(1)] \\ \hat{u}(0) &= 0 \\ \hat{u}_x(1) &= u_x(1) - \frac{\lambda}{2}[u(1) - \hat{u}(1)] \end{aligned}$$

Controller

$$u_x(1) = -\frac{\lambda}{2} u(1) - \int_0^1 \frac{\lambda y}{1-y^2} I_2\left(\sqrt{\lambda(1-y^2)}\right) \hat{u}(y)\, dy$$

Frequency Domain Representation

Plant

$$\begin{aligned} u_t &= u_{xx} + gu(0) && (17) \\ u_x(0) &= 0 && (18) \end{aligned}$$

Output: $u(0)$ Input: $u(1)$

To derive the transfer function from $u(1)$ to $u(0)$, take the Laplace transform of the plant:

$$\begin{aligned} su(x,s) &= u''(x,s) + gu(0,s) && (19) \\ u'(0,s) &= 0 && (20) \end{aligned}$$

General solution for this ODE:

$$u(x,s) = A\sinh(\sqrt{s}x) + B\cosh(\sqrt{s}x) + \frac{g}{s}u(0,s) \tag{21}$$

and boundary condition gives

$$u'(0,s) = A\sqrt{s} = 0 \Rightarrow A = 0 \tag{22}$$

Backstepping Observer for Parabolic PDEs: Design $\rightarrow$ Discretize

We have

$$u(x,s) = B\cosh(\sqrt{s}x) + \frac{g}{s}u(0,s) \tag{23}$$

Setting $x = 0$ we get

$$B = u(0,s)\left(1 - \frac{g}{s}\right) \tag{24}$$

so that

$$u(x,s) = u(0,s)\left[\frac{g}{s} + \left(1 - \frac{g}{s}\right)\cosh(\sqrt{s}x)\right] \tag{25}$$

Setting $x = 1$ we get the open-loop transfer function

$$u(0,s) = \frac{s}{g + (s-g)\cosh(\sqrt{s})}u(1,s) \tag{26}$$

There are infinitely many poles and no zeros (infinite relative degree). This can be seen by replacing the $\cosh$ term by its Taylor series expansion.

Let us now design compensator. The observer is

$$\begin{aligned}\hat{u}_t &= \hat{u}_{xx} + gu(0)\\ \hat{u}_x(0) &= 0\\ \hat{u}(1) &= -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))\hat{u}(y)dy\end{aligned}$$

Applying Laplace transform we get

$$\begin{aligned}s\hat{u}(x,s) &= \hat{u}''(x,s) + gu(0,s)\\ \hat{u}'(0,s) &= 0\\ \hat{u}(1,s) &= -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))\hat{u}(y,s)dy\end{aligned}$$

The solution of this ODE is

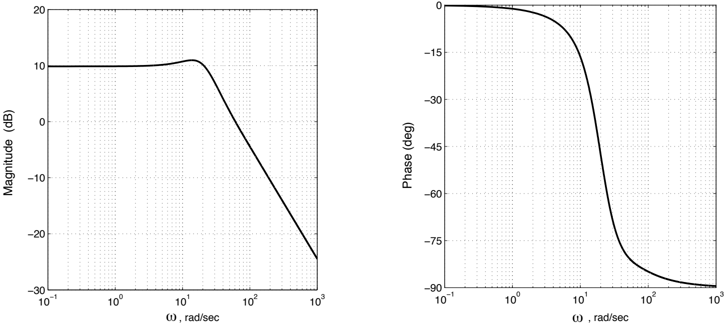
$$\hat{u}(x,s) = \hat{u}(0,s)\cosh(\sqrt{s}x) + \frac{g}{s}\left(1-\cosh(\sqrt{s}x)\right)u(0,s)$$

Setting $x=1$ and using the boundary condition we express $\hat{u}(0,s)$ as a function of $u(0,s)$:

$$\hat{u}(0,s) = \frac{\cosh(\sqrt{s}) - \cosh(\sqrt{g})}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})}gu(0,s)$$

Finally, the compensator is

$$u(1,s)=\frac{g}{s}\left(-1+\frac{(s-g)\cosh(\sqrt{s}\cosh(\sqrt{g}))}{s\cosh(\sqrt{s})-g\cosh(\sqrt{g})}\right)u(0,s)$$



Approximation:

$$u(1,s)\approx 60\frac{s+17}{s^2+25s+320}u(0,s)$$