Control of PDE Systems

Lecture 6 (Meetings 11 & 12)

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Backstepping Control Design

Unstable heat equation

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(0) &= 0 \\ u(1) &= \text{control} \end{aligned}$$

Backstepping transformation

$$w(x) = u(x) - \int_0^{x} k(x,y)u(y)\,dy$$

Target system

$$\begin{aligned} w_t &= w_{xx} \\ w(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Controller is obtained by setting $x = 1$ in the transformation

$$u(1) = \int_0^{1} k(1,y)u(y)\,dy$$

Useful knowledge from calculus: Leibniz Integral Rule

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} f_z(x,z)dx + f(b(z),z)b'(z) - f(a(z),z)a'(z)$$

Notation:

$$\begin{aligned}
k_x(x,x) &= \frac{\partial}{\partial x}k(x,y)|_{y=x} \\
k_y(x,x) &= \frac{\partial}{\partial y}k(x,y)|_{y=x} \\
\frac{d}{dx}k(x,x) &= k_x(x,x) + k_y(x,x)
\end{aligned}$$

A particular case of the Leibnitz differentiation rule, which is more related to our proposed Volterra integral transformation, is the following:

$$\frac{d}{dx}\int_0^x f(x,y)dy = f(x,x) + \int_0^x f_x(x,y)dy$$

Kernel PDE Derivation

$$
\begin{aligned}
w(x) &= u(x) - \int_0^x k(x,y)u(y)dy \\
w_x(x) &= u_x(x) - \int_0^x k_x(x,y)u(y)dy - k(x,x)u(x) \\
w_{xx}(x) &= u_{xx}(x) - \int_0^x k_{xx}(x,y)u(y)dy - k_x(x,x)u(x) - \frac{d}{dx}(k(x,x)u(x))
\end{aligned}
$$

Time derivative:

$$
\begin{aligned}
w_t(x) &= u_t(x) - \int_0^x k(x,y)u_t(y)dy \\
&= u_{xx}(x) + \lambda u(x) - \int_0^x k(x,y)\left[u_{yy}(y) + \lambda u(y)\right]dy \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + \int_0^x k_y(x,y)u_y(y)dy \\
&\quad - \int_0^x \lambda k(x,y)u(y)dy \quad \text{(integration by parts)} \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)dy - \int_0^x \lambda k(x,y)u(y)dy \quad \text{(integration by parts)}
\end{aligned}
$$

Backstepping Control of Parabolic PDEs: Design $\rightarrow$ Discretize

$$
\begin{aligned}
w_t - w_{xx} &= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy \\
&\quad - \left[u_{xx}(x) - \int_0^x k_{xx}(x,y)u(y)\,dy - k_x(x,x)u(x) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x)) \right] \\
&= u(x)\left[\lambda + 2\frac{d}{dx}k(x,x)\right] + k(x,0)u_x(0) \\
&\quad + \int_0^x u(y)\left[k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)\right] dy
\end{aligned}
$$

For right hand side to be zero, 3 conditions should be satisfied:

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y) \\
k(x,0) &= 0 \\
\lambda + 2\frac{d}{dx}k(x,x) &= 0
\end{aligned}
$$

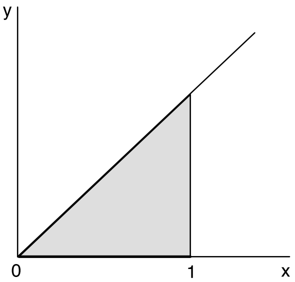
Are these 3 conditions compatible? In other words, is this PDE well posed?

Backstepping Control of Parabolic PDEs: Design → Discretize

Control kernel PDE

$$\begin{aligned} k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y) \\ k(x,0) &= 0 \\ k(x,x) &= -\frac{\lambda x}{2} \end{aligned}$$

Domain



Converting Kernel PDE to Integral Equation

Introduce the change of variables

$$\begin{aligned} \xi &= x+y \\ \eta &= x-y \\ k(x,y) &= G(\xi,\eta) \end{aligned}$$

Then we have

$$\begin{aligned} k_x &= G_\xi + G_\eta \\ k_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} \\ k_y &= G_\xi - G_\eta \\ k_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta} \end{aligned}$$

The kernel PDE in new variables is

$$\begin{aligned} 4G_{\xi\eta}(\xi,\eta) &= \lambda G(\xi,\eta) \\ G(\xi,\xi) &= 0 \\ G(\xi,0) &= -\frac{\lambda\xi}{4} \end{aligned}$$

Integrate $4G_{\xi\eta} = \lambda G$ with respect to η from 0 to η:

$$G_\xi(\xi,\eta) = G_\xi(\xi,0) + \int_0^\eta \frac{\lambda}{4} G(\xi,s)\,ds$$

Integrate the result with respect to ξ from η to ξ and use boundary conditions to get

$$G(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G(\tau,s)\,ds\,d\tau$$

How to solve this integral equation?

Method of Successive Approximations

Very simple idea: start with a guess, compute the right hand side of the equation, use the solution as the next guess and repeat. The result will converge to the solution of the integral equation.

Let us start with initial guess

$$G_0(\xi,\eta) = 0 \tag{1}$$

and define the recursive formula as

$$G_{n+1}(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G_n(\tau,s)\,ds\,d\tau \tag{2}$$

If this converges, we can write the solution $G(\xi,\eta)$ as

$$G(\xi,\eta) = \lim_{n\to\infty} G_n(\xi,\eta)$$

Let us denote the difference between two consecutive terms as

$$\Delta G_{n+1}(\xi,\eta) = G_{n+1}(\xi,\eta) - G_n(\xi,\eta) \tag{3}$$

Then,

$$\Delta G_{n+1}(\xi,\eta) = \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} \Delta G_n(\tau,s)dsd\tau \tag{4}$$

And we can alternatively write

$$G(\xi,\eta) = \lim_{n\to\infty} G_n(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G_{n+1}(\xi,\eta) \quad \text{(recall that } G_0(\xi,\eta)=0\text{)}$$

Note from (1) and (3) that

$$\Delta G_1(\xi,\eta) = G_1(\xi,\eta) \tag{5}$$

Moreover, from (2)

$$\Delta G_1(\xi,\eta) = G_1(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) \tag{6}$$

Now, from (4) we can write

$$\Delta G_2(\xi,\eta) = \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta \Delta G_1(\tau,s)dsd\tau = \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta \left(-\frac{\lambda}{4}(\tau - s)\right)dsd\tau$$

Then,

$$\begin{aligned}
\Delta G_2(\xi,\eta) &= \left(\frac{\lambda}{4}\right)^2 \int_\eta^\xi \int_0^\eta (s-\tau)dsd\tau \\
&= \left(\frac{\lambda}{4}\right)^2 \int_\eta^\xi \frac{1}{2}(s-\tau)^2|_0^\eta d\tau \\
&= \left(\frac{\lambda}{4}\right)^2 \int_\eta^\xi \frac{1}{2}\left[(\eta-\tau)^2 - \tau^2\right]d\tau \\
&= \left(\frac{\lambda}{4}\right)^2 \frac{1}{2}\frac{1}{3}\left[(\tau-\eta)^3 - \tau^3\right]|_\eta^\xi \\
&= \left(\frac{\lambda}{4}\right)^2 \frac{1}{2}\frac{1}{3}\left[(\xi-\eta)^3 - (\xi^3-\eta^3)\right]
\end{aligned}$$

Noting that

$$(\xi-\eta)^3 = \xi^3 - 3\eta\xi^2 + 3\eta^2\xi - \eta^3$$

Then,

$$(\xi-\eta)^3 - (\xi^3 - \eta^3) = -3\eta\xi^2 + 3\eta^2\xi = -3\eta\xi(\xi-\eta)$$

And finally,

$$\Delta G_2(\xi,\eta) \quad = \quad -\left(\frac{\lambda}{4}\right)^2 \frac{1}{2}\eta\xi(\xi-\eta)$$

By repeating this process it is possible to observe a pattern that leads to

$$\Delta G_{n+1}(\xi,\eta) = -\left(\frac{\lambda}{4}\right)^{n+1} \frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!} \tag{7}$$

and we can write

$$G(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G_{n+1}(\xi,\eta) = -\sum_{n=0}^{\infty} \frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{\lambda}{4}\right)^{n+1} \tag{8}$$

Backstepping Control of Parabolic PDEs: Design → Discretize

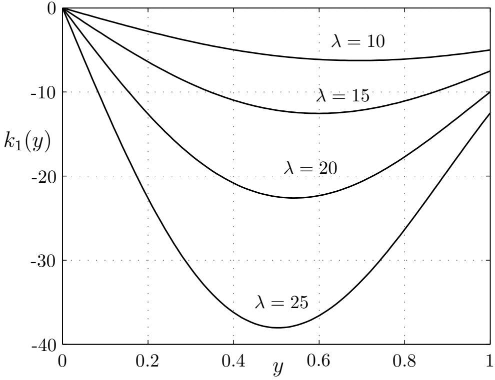
This series can be summed up:

$$G(\xi,\eta) = -\frac{\lambda}{2}(\xi-\eta)\frac{I_1\left(\sqrt{\lambda\xi\eta}\right)}{\sqrt{\lambda\xi\eta}}$$

or in the original variables

$$k(x,y) = -\lambda y\frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}$$

Control Gain



Bessel Functions J_n and I_n

The function $y(x) = J_n(x)$ is a solution to the following ODE

$$x^2 y''_{xx} + x y'_x + (x^2 - n^2) y = 0$$

Series representation

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!}$$

Other properties

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x))$$

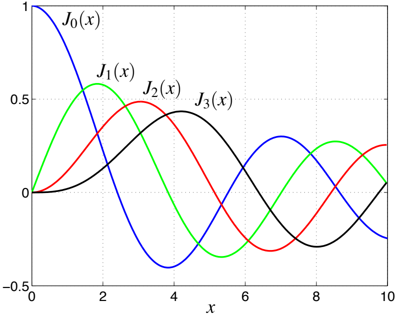
Differentiation

$$\frac{d}{dx} J_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) \qquad \frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}$$

Asymptotic properties

$$J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right), \qquad x \to \infty$$



The function $y(x) = I_n(x)$ is a solution to the following ODE

$$x^2 y''_{xx} + x y'_x - (x^2 + n^2) y = 0$$

Series representation

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!}$$

Relationship with $J_n(x)$

$$I_n(x) = i^{-n} J_n(ix), \qquad I_n(ix) = i^n J_n(x)$$

Other properties

$$2nI_n(x) = x(I_{n-1}(x) - I_{n+1}(x))$$

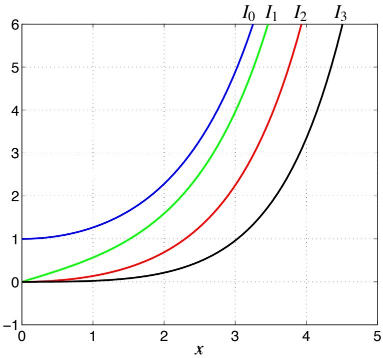
Differentiation

$$\frac{d}{dx} I_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) \qquad \frac{d}{dx}(x^n I_n(x)) = x^n I_{n-1}$$

Asymptotic properties

$$I_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0$$

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \qquad x \to \infty$$



Inverse Transformation

Remember the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)dy$$

We found $k(x,y)$ and w-system is exp. stable. Does this imply that u is exp. stable?

Depends on the properties of $k(x,y)$. Since our kernel $k(x,y)$ is twice continuously differentiable, it turns out that this is enough for inverse transformation to exist.

Let us find the inverse transformation

$$u(x) = w(x) + \int_0^x l(x,y)w(y)dy$$

It can be shown that $l(x,y)$ satisfies the following PDE

$$\begin{aligned} l_{xx}(x,y) - l_{yy}(x,y) &= -\lambda l(x,y) \\ l(x,0) &= 0 \qquad\qquad \Rightarrow \quad l(\lambda) = -k(-\lambda)! \\ l(x,x) &= -\frac{\lambda x}{2} \end{aligned}$$

We have

$$\begin{aligned} l(x,y) &= (-\lambda)y\frac{I_1\left(\sqrt{(-\lambda)(x^2-y^2)}\right)}{\sqrt{-\lambda(x^2-y^2)}} = -\lambda y\frac{I_1\left(j\sqrt{\lambda(x^2-y^2)}\right)}{j\sqrt{\lambda(x^2-y^2)}} \\ &= -\lambda y\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \end{aligned}$$

Therefore the inverse transformation is

$$u(x) = w(x) - \int_0^x \lambda y\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}w(y)\,dy$$

Since $w(x,t) \to 0$ as $t \to \infty$, we get $u(x,t) \to 0$ for all $x \in [0,1]$ with a boundary controller

$$u(1) = -\int_0^1 \lambda y\frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}u(y)\,dy$$

Notes: How does the PDE system for $l(x, y)$ is obtained?

Let us first differentiate the transformation w.r.t. time, i.e.

$$
\begin{aligned}
u_t(x) &= w_t(x) + \int_0^x l(x,y) w_t(y) dy \\
&= w_{xx}(x) + \int_0^x l(x,y) w_{yy}(y) dy \\
&= w_{xx}(x) + l(x,y) w_x(y)|_0^x - \int_0^x l_y(x,y) w_y(y) dy \text{ (integration by parts)} \\
&= w_{xx}(x) + l(x,x) w_x(x) - l(x,0) w_x(0) \\
&\quad -l_y(x,y) w(y)|_0^x + \int_0^x l_{yy}(x,y) w(y) dy \text{ (integration by parts)} \\
&= w_{xx}(x) + l(x,x) w_x(x) - l(x,0) w_x(0) \\
&\quad -l_y(x,x) w(x) + \int_0^x l_{yy}(x,y) w(y) dy \quad (w(0) \equiv 0)
\end{aligned}
$$

Let us now differentiate the transformation w.r.t. space, i.e.

$$
\begin{aligned}
u_x(x) &= w_x(x) + \int_0^x l_x(x,y)w(y)dy + l(x,x)w(x) \text{ (Leibnitz Rule)} \\
u_{xx}(x) &= w_{xx}(x) + \frac{d}{dx}\left(l(x,x)w(x)\right) \\
&+ \int_0^x l_{xx}(x,y)w(y)dy + l_x(x,x)w(x) \text{ (Leibnitz Rule)} \\
u_{xx}(x) &= w_{xx}(x) + l(x,x)w_x(x) + w(x)\frac{d}{dx}l(x,x) + l_x(x,x)w(x) \\
&+ \int_0^x l_{xx}(x,y)w(y)dy
\end{aligned}
$$

Then,

$$
\begin{aligned}
u_t - u_{xx} = \lambda u &= \lambda w + \lambda \int_0^x l(x,y)w(y)dy \\
&= -l(x,0)w_x(0) - 2w(x)\frac{d}{dx}l(x,x) \\
&+ \int_0^x \left(l_{yy}(x,y) - l_{xx}(x,y)\right)w(y)dy
\end{aligned}
$$

This leads to the following conditions

$$\begin{aligned} l_{yy}(x,y) - l_{xx}(x,y &= \lambda l(x,y) \\ l(x,0) &= 0 \\ \frac{d}{dx} l(x,x) &= -\frac{\lambda}{2} \end{aligned}$$

Backstepping Control of Parabolic PDEs: Design → Discretize

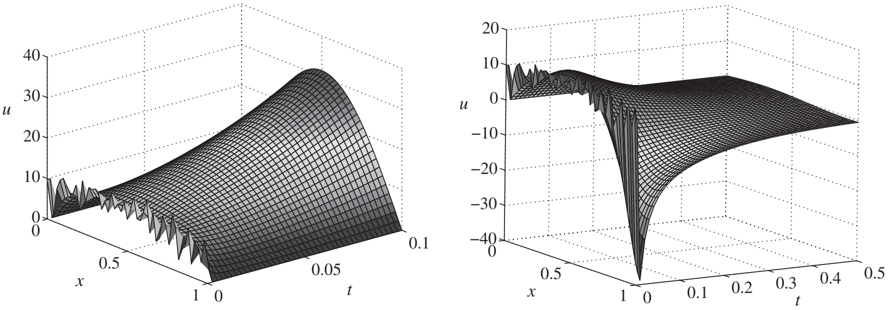


Figure 1: Left: Open-loop. Right: Closed-loop.

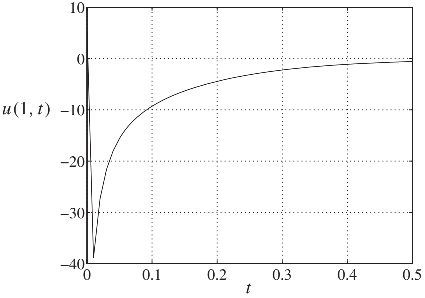


Figure 2: Control at $x = 1$.

$$u(1) = \int_0^1 k(1, y)u(y)dy = -\lambda \int_0^1 y \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} u(y)dy \qquad (9)$$

Neumann Controller

Unstable heat equation with Neumann actuation

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(0) &= 0 \\ u_x(1) &= \text{control} \end{aligned}$$

Exactly the same transformation as in case of Dirichlet actuation:

$$w(x) = u(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} u(y)\, dy$$

But with a target system modified at $x = 1$ (easy to show that it is stable)

$$\begin{aligned} w_t &= w_{xx} \\ w(0) &= 0 \\ w_x(1) &= 0 \end{aligned}$$

Notes: How do we obtain the control $u_x(1)$?

We need to differentiate the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)dy$$

w.r.t. space, i.e.

$$w_x(x) = u_x(x) - \int_0^x k_x(x,y)u(y)dy - k(x,x)u(x) \text{ (Leibnitz Rule)}$$

and set $x = 1$ (it is clear now why the target system has been chosen with an homogeneous Neumann boundary condition at $x = 1$) to obtain

$$u_x(1) = k(1,1)u(1) + \int_0^1 k_x(1,y)u(y)dy$$

with

$$k(x,y) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}}$$

Backstepping Control of Parabolic PDEs: Design $\rightarrow$ Discretize

Simply differentiate the transformation with respect to x:

$$w(x) = u(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}} u(y)\, dy$$

$$w_x(x) = u_x(x) + \frac{\lambda x}{2} u(x) + \int_0^x \lambda y x \frac{I_2\left(\sqrt{\lambda(x^2 - y^2)}\right)}{x^2 - y^2} u(y)\, dy$$

and evaluate at $x = 1$ to get Neumann controller:

$$u_x(1) = -\frac{\lambda}{2} u(1) - \int_0^1 \lambda y \frac{I_2\left(\sqrt{\lambda(1 - y^2)}\right)}{1 - y^2} u(y) dy$$

Notes: How do we compute $k_x(x,y)$?

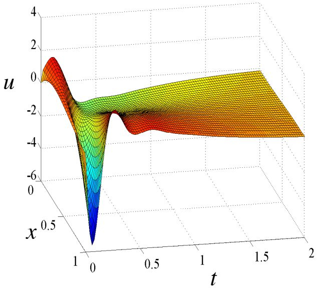
We start from

$$k(x,y) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}$$

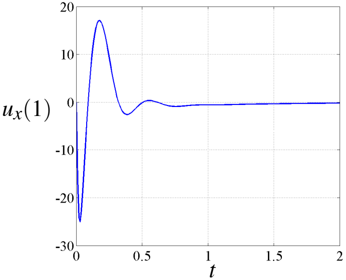
and compute

$$\begin{aligned}
k_x(x,y) &= -\lambda y \left[\frac{\frac{d}{dx} I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}} - \frac{1}{2}(2\lambda x) \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{(\lambda(x^2-y^2))^{3/2}} \right] \\
&= -\lambda y \left[\frac{\left(\frac{1}{\sqrt{\lambda(x^2-y^2)}} I_1(\sqrt{\lambda(x^2-y^2)}) + I_2(\sqrt{\lambda(x^2-y^2)}) \right) \frac{(2\lambda x)}{2\sqrt{\lambda(x^2-y^2)}}}{\sqrt{\lambda(x^2-y^2)}} \right. \\
&\qquad \left. - \frac{1}{2}(2\lambda x) \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{(\lambda(x^2-y^2))^{3/2}} \right] \\
&= -\lambda y x \frac{I_2(\sqrt{\lambda(x^2-y^2)})}{x^2-y^2}
\end{aligned}$$

Closed Loop Simulation



Closed loop state



Control effort

Reaction-Advection-Diffusion Systems

Plant

$$
\begin{aligned}
u_t &= \epsilon u_{xx} + b u_x + \lambda u && (10)\\
u(0) &= 0 && (11)\\
u(1) &= U(1) && (12)
\end{aligned}
$$

Let us define the change of variable $v(x) = u(x) e^{\frac{b}{2\epsilon}x}$. Then,

$$
\begin{aligned}
u_t &= v_t(x) e^{-\frac{b}{2\epsilon}x} && (13)\\
u_x &= v_x(x) e^{-\frac{b}{2\epsilon}x} - \frac{b}{2\epsilon} v(x) e^{-\frac{b}{2\epsilon}x} && (14)\\
u_{xx} &= v_{xx}(x) e^{-\frac{b}{2\epsilon}x} - \frac{b}{\epsilon} v_x(x) e^{-\frac{b}{2\epsilon}x} + \frac{b^2}{4\epsilon^2} v(x) e^{-\frac{b}{2\epsilon}x} && (15)
\end{aligned}
$$

and

$$
v_t(x) e^{-\frac{b}{2\epsilon}x} = \left(\epsilon v_{xx}(x) - b v_x(x) + \frac{b^2}{4\epsilon} v(x) + b v_x(x) - \frac{b^2}{2\epsilon} v(x) + \lambda v(x) \right) e^{-\frac{b}{2\epsilon}x}
$$

The plant can be rewritten as

$$\begin{aligned} v_t &= \epsilon v_{xx} + \left(\lambda - \frac{b^2}{4\epsilon}\right) v && (16) \\ v(0) &= 0 && (17) \\ v(1) &= u(1)e^{\frac{b}{2\epsilon}} = \text{ control} && (18) \end{aligned}$$

Now we can follow the procedure introduced for the original (Dirichlet) problem.

$$w(x) = v(x) - \int_0^x k(x,y)v(y)dy$$

Target

$$\begin{aligned} w_t &= \epsilon w_{xx} - cw && (19) \\ w(0) &= 0 && (20) \\ w(1) &= 0 && (21) \end{aligned}$$

with $c \geq \max\left\{\frac{b^2}{4\epsilon} - \lambda, 0\right\}$ → decay rate closed-loop system (no effort if stable).

The gain kernel $k(x, y)$ can be shown to satisfy

$$\epsilon k_{xx}(x,y) - \epsilon k_{yy}(x,y) = \left(\lambda - \frac{b^2}{4\epsilon} + c\right) k(x,y) \quad (22)$$

$$k(x,0) = 0 \quad (23)$$

$$k(x,x) = -\frac{x}{2\epsilon}\left(\lambda - \frac{b^2}{4\epsilon} + c\right) \quad (24)$$

This is the same PDE system as before with $\lambda_0 = \frac{1}{\epsilon}\left(\lambda - \frac{b^2}{4\epsilon} + c\right)$ and solution

$$k(x,y) = -\lambda_0 y \frac{I_1(\sqrt{\lambda_0(x^2 - y^2)})}{\sqrt{\lambda_0(x^2 - y^2)}}$$

The controller is given by

$$u(1) = \int_0^1 e^{-\frac{b}{2\epsilon}(1-y)} \lambda_0 y \frac{I_1(\sqrt{\lambda_0(1 - y^2)})}{\sqrt{\lambda_0(1 - y^2)}} u(y) dy$$

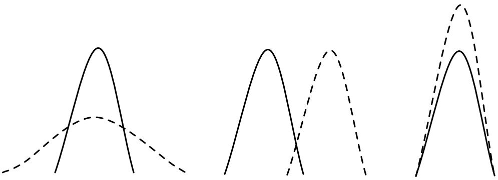
Reaction-Advection-Diffusion Systems

Plant

$$
\begin{aligned}
u_t &= \varepsilon(x)u_{xx} + b(x)u_x + \lambda(x)u \\
u_x(0) &= -qu(0)
\end{aligned}
$$

These equations come from thermal / fluid / chemical problems.

What each term does:



Diffusion (smoothing) — Advection (transfer) — Reaction (destabilizing)

Reaction-Advection-Diffusion Systems

Plant

$$\begin{aligned} u_t &= \varepsilon(x)u_{xx} + b(x)u_x + \lambda(x)u \\ u_x(0) &= -qu(0) \end{aligned}$$

Spatially varying coefficients arise for several reasons:

— linearization

— non-homogenous materials

— unusually shaped domains

Using special transformation we can eliminate $b(x)$ and make $\varepsilon(x)$ constant.

Gauge Transformation (→ constant diffusion, zero advection)
Coordinate change:

$$z = \sqrt{\epsilon_0} \int_0^x \frac{ds}{\sqrt{\epsilon(s)}}, \quad \epsilon_0 = \left(\int_0^1 \frac{ds}{\sqrt{\epsilon(s)}} \right)^{-2} \tag{25}$$

State-variable change:

$$v(z) = \epsilon^{-1/4}(x) u(x) e^{\int_0^x \frac{b(s)}{2\epsilon(s)} ds} \tag{26}$$

It is possible to show that the new state variable satisfies

$$v_t(z,t) = \epsilon_0 v_{zz}(z,t) + \lambda_0(z) v(z,t) \tag{27}$$

$$v_z(0,t) = -q_0 v(0,t) \tag{28}$$

where

$$\lambda_0(z) = \lambda(x) + \frac{\epsilon''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16}\frac{(\epsilon'(x))^2}{\epsilon(x)} + \frac{1}{2}\frac{b(x)\epsilon'(x)}{\epsilon(x)} - \frac{1}{4}\frac{b^2(x)}{\epsilon(x)} \tag{29}$$

$$q_0 = q\sqrt{\frac{\epsilon(0)}{\epsilon_0}} - \frac{b(0)}{2\sqrt{\epsilon_0 \epsilon(0)}} - \frac{\epsilon'(0)}{4\sqrt{\epsilon_0 \epsilon(0)}} \tag{30}$$

We use the transformation

$$w(x) = v(x) - \int_0^x k(x,y)v(y)dy \tag{31}$$

to map the modified plant into the target system

$$\begin{aligned} w_t &= \epsilon_0 w_{xx} - cw & (32)\\ w_z(0) &= 0 & (33)\\ w(1) &= 0) & (34) \end{aligned}$$

The constant c is a design parameter that determines the decay rate of the closed-loop system. The transformation kernel is found by solving the PDE

$$\begin{aligned} k_{zz}(z,y) - k_{yy}(z,y) &= \frac{\lambda_0(y)+c}{\epsilon_0}k(z,y) & (35)\\ k_y(z,0) &= -q_0 k(z,0) & (36)\\ k(z,z) &= -q_0 - \frac{1}{2\epsilon_0}\int_0^z (\lambda_0(y)+c)dy & (37) \end{aligned}$$

This kernel PDE can no longer be solved in closed form, but the solution can be computed numerically (order of magnitude faster in computation time than solving a Riccati equation).

Since the controller for the v-system is given by

$$v(1) = \int_0^1 k(1,y)v(y)dy, \tag{38}$$

using both the coordinate change and the state-variable change we can write is as

$$u(1) = \int_0^1 \frac{\epsilon^{1/4}(1)\sqrt{\epsilon_0}}{\epsilon^{3/4}(y)} e^{-\int_y^1 \frac{b(s)}{2\epsilon(s)}ds} k\left(\int_0^1 \sqrt{\frac{\epsilon_0}{\epsilon(s)}}ds, \int_0^y \sqrt{\frac{\epsilon_0}{\epsilon(s)}}ds\right) u(y)dy. \tag{39}$$