Control of PDE Systems

Lecture 5 (Meetings 9 & 10)

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Backstepping Control of Parabolic PDEs: Discretize → Design

Let us consider the following parabolic PDE [1]

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial E}{\partial r}\right] + P, \qquad (1)$$

with arbitrary boundary conditions

$$\left.\frac{\partial E}{\partial r}\right|_{r=0} = 0, \qquad (2)$$

$$\left.\frac{\partial E}{\partial r}\right|_{r=a} = k_E E(a). \qquad (3)$$

Note that $E = E(r,t)$, $D = D(E(r,t))$ and $P = P(r)$.

[1] The paper on "Control of a non-linear PDE system arising from non-burning tokamak plasma transport dynamics" by Schuster et al. considers a rather more complex system where the energy equation is combined with the density equation.

Backstepping Control of Parabolic PDEs: Discretize → Design

We write $E(r,t) = \bar{E}(r) + \tilde{E}(r,t)$, where $\bar{E}(r)$ is the equilibrium profile which satisfies the equilibrium equation

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\bar{D}\frac{\partial \bar{E}}{\partial r}\right] + \bar{P}, \tag{4}$$

with boundary conditions

$$\left.\frac{\partial \bar{E}}{\partial r}\right|_{r=0} = 0, \tag{5}$$

$$\left.\frac{\partial \bar{E}}{\partial r}\right|_{r=a} = k_E \bar{E}(a), \tag{6}$$

Note that the equilibrium profile will depend not only on the boundary conditions but also on the interior source $\bar{P}$.

It is important to note that in this approach we consider only boundary actuation. Therefore, $P = \bar{P}$ is used only for the definition of the equilibrium profile.

The dynamics of the deviation variables $\tilde{E}(r,t)$ is given by

$$\begin{aligned} \frac{\partial \tilde{E}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial(\bar{E}+\tilde{E})}{\partial r}\right] + P \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \bar{E}}{\partial r}\right] + P. \end{aligned}$$

We take into account that

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial(\cdot)}{\partial r}\right] = \frac{\partial}{\partial r}\left[D\frac{\partial(\cdot)}{\partial r}\right] + \frac{1}{r}D\frac{\partial(\cdot)}{\partial r},$$

and we define

$$g(E) = \frac{\partial}{\partial r}\left[D\frac{\partial \bar{E}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \bar{E}}{\partial r} + P. \tag{7}$$

In this way we can rewrite the equations for the deviation variables as

$$\frac{\partial \tilde{E}}{\partial t} = \frac{\partial}{\partial r}\left[D\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{E}}{\partial r} + g(E), \quad (8)$$

with boundary conditions

$$\left.\frac{\partial \tilde{E}}{\partial r}\right|_{r=0} = 0, \quad (9)$$

$$\left.\frac{\partial \tilde{E}}{\partial r}\right|_{r=a} = k_E \tilde{E}(a) + \Delta \tilde{E}_r, \quad (10)$$

The objective is to stabilize $\tilde{E}(r,t)$, making it converge to zero, by using $\Delta\tilde{E}_r(t)$ as actuation at the edge of the domain.

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

- We dicretize the original set of PDE's in space using a finite difference method which gives a high order set of coupled nonlinear ordinary differential equations.

- We apply backstepping design to obtain a discretized coordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. To achieve such stability for the target system, convenient boundary conditions are chosen.

- We use the property that the discretized coordinate transformation is invertible for an arbitrary (finite) grid choice to conclude that the discretized version of the original system is asymptotically stable and we obtain a nonlinear feedback boundary control law in the original set of coordinates.

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

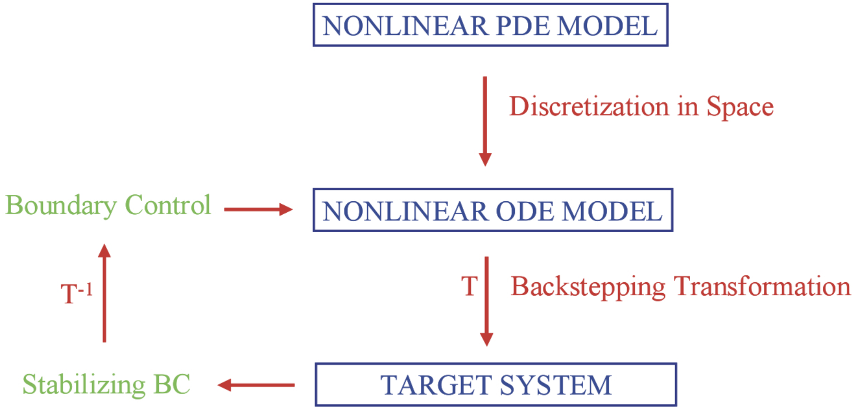

Figure 1: Control Approach

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

- The idea is to design a controller using only a small number of steps of backstepping, or equivalently using only a small number of state measurements.
- The measurements are taken from the interior of the domain and the actuation is applied at the boundary of the domain.
- To discretize the problem, let us start by defining $h = \frac{1}{N}$, where N is an integer, and use the notation $x_i(t) = x(ih, t)$, $i = 0, 1, ..., N$.

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

Discretization Method:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial x}{\partial r}\right] \;=\; \frac{\partial}{\partial r}\left[D\frac{\partial x}{\partial r}\right]+\frac{1}{r}D\frac{\partial x}{\partial r}$$

First-order derivative:

$$\left.\frac{1}{r}D\frac{\partial x}{\partial r}\right|_{i}=\frac{1}{ih}D_i\frac{x_{i+1}-x_i}{h}$$

Second-order derivative:

$$\left.\frac{\partial x}{\partial r}\right|_{i+\frac{1}{2}} \;=\; \frac{x_{i+1}-x_i}{h}$$

$$\left.\frac{\partial x}{\partial r}\right|_{i-\frac{1}{2}} \;=\; \frac{x_i-x_{i-1}}{h}$$

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$$
\begin{aligned}
\frac{\partial}{\partial r}\left[D\frac{\partial x}{\partial r}\right]_i &= \frac{D_{i+\frac{1}{2}}\ \frac{\partial x}{\partial r}\Big|_{i+\frac{1}{2}} - D_{i-\frac{1}{2}}\ \frac{\partial x}{\partial r}\Big|_{i-\frac{1}{2}}}{h} \\
&= \frac{D_{i+\frac{1}{2}}\left(\frac{x_{i+1}-x_i}{h}\right) - D_{i-\frac{1}{2}}\left(\frac{x_i-x_{i-1}}{h}\right)}{h} \\
&= \frac{D_{i+\frac{1}{2}}x_{i+1} - \left(D_{i-\frac{1}{2}} + D_{i+\frac{1}{2}}\right)x_i + D_{i-\frac{1}{2}}x_{i-1}}{h^2} \\
&= \frac{D_{i+\frac{1}{2}}x_{i+1} - 2D_ix_i + D_{i-\frac{1}{2}}x_{i-1}}{h^2}
\end{aligned}
$$

$$
\begin{aligned}
D_{i-\frac{1}{2}} &= D_i - \frac{D_i-D_{i-1}}{h}\frac{h}{2} = \frac{1}{2}D_i + \frac{1}{2}D_{i-1} \\
D_{i+\frac{1}{2}} &= D_i + \frac{D_i-D_{i-1}}{h}\frac{h}{2} = \frac{3}{2}D_i - \frac{1}{2}D_{i-1}
\end{aligned}
\Rightarrow D_i = \frac{D_{i+\frac{1}{2}} + D_{i-\frac{1}{2}}}{2}
$$

We write the discretized version of equation (8) as

$$\dot{\tilde{E}}_i = \frac{D_{i+\frac{1}{2}}\tilde{E}_{i+1} - 2D_i\tilde{E}_i + D_{i-\frac{1}{2}}\tilde{E}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{E}_{i+1} - \tilde{E}_i}{h} + g_i, \quad (11)$$

$$g_i = \frac{D_{i+\frac{1}{2}}\bar{E}_{i+1} - 2D_i\bar{E}_i + D_{i-\frac{1}{2}}\bar{E}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\bar{E}_{i+1} - \bar{E}_i}{h} + \bar{P}_i,$$

for $i = 1, ..., N-1$ and the discretized version of the boundary condition equations (9)-(10) as

$$\frac{\tilde{E}_1 - \tilde{E}_0}{h} = 0, \quad (12)$$

$$\frac{\tilde{E}_N - \tilde{E}_{N-1}}{h} = k_E\tilde{E}_N + \Delta\tilde{E}_r, \quad (13)$$

where

$$D_{i-\frac{1}{2}} = D_i - \frac{D_i - D_{i-1}}{h}\frac{h}{2} = \frac{1}{2}D_i + \frac{1}{2}D_{i-1},$$

$$D_{i+\frac{1}{2}} = D_i + \frac{D_i - D_{i-1}}{h}\frac{h}{2} = \frac{3}{2}D_i - \frac{1}{2}D_{i-1}.$$

- Note that the system (11)–(13) is a general **Strict Feedback System**, i.e., it can be written as

$$\begin{aligned} \dot{x}_i &= \psi_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= \psi_n(x)u + \phi_n(x) \end{aligned}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T$ ($\bar{x}_n = x$), $\phi_i(\bar{x}_i)$ are smooth and $\phi_i(0) = 0$, and $\psi_i(\bar{x}_i) \neq 0$ for $i = 1, \ldots, n$ over the domain of interest.

- The choice of a backward approximation for the derivatives of D at point i is key to our approach. In this way it is possible to write $D_{i+\frac{1}{2}}$ as functions of the state variables at points i and $i-1$. Otherwise, the system would NOT be strict feedback.

We already studied how to design a backstepping controller for this type of system. Do you remember?

$$
\begin{aligned}
\dot{x}_i &= x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n-1 \\
\dot{x}_n &= u + \phi_n(x)
\end{aligned}
$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T$, $\phi_i(\bar{x}_i)$ are smooth and $\phi_i(0) = 0$.
We have a local triangular structure:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \phi_1(x_1) \\
\dot{x}_2 &= x_3 + \phi_2(x_1, x_2) \\
&\vdots \\
\dot{x}_n &= u + \phi_n(x_1, x_2, \ldots, x_n)
\end{aligned}
$$

Linear part: Brunovsky canonical form $\Rightarrow$ feedback linearizable

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

The control law

$$
\begin{aligned}
z_i &= x_i - \alpha_{i-1}(\bar{x}_{i-1}) \quad \alpha_0 = 0 \\
\alpha_i(\bar{x}_i) &= -z_{i-1} - c_i z_i - \phi_i + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j), \quad c_i > 0 \\
u &= \alpha_n
\end{aligned}
$$

guarantees global asymptotic stability of $x = 0$.

- We consider now the asymptotically stable (in L^2 norm) target system

$$\begin{aligned} \frac{\partial \tilde{F}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C_F\tilde{F} \\ &= \frac{\partial}{\partial r}\left[D\frac{\partial \tilde{F}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{F}}{\partial r} - C_F\tilde{F}, \end{aligned} \tag{14}$$

where $C_F > 0$ and the boundary conditions given by $(G > 0)$

$$\left.\frac{\partial \tilde{F}}{\partial r}\right|_{r=0} = 0, \tag{15}$$

$$\left.\frac{\partial \tilde{F}}{\partial r}\right|_{r=a} = -G\tilde{F}(a). \tag{16}$$

- The choice of the target system is based on the need to maintain the parabolic character of the partial differential equation (to keep the highest order derivatives) while removing the "problematic" terms.

- We write the discretized equations for the target system as

$$\dot{\tilde{F}}_i = \frac{D_{i+\frac{1}{2}}\tilde{F}_{i+1} - 2D_i\tilde{F}_i + D_{i-\frac{1}{2}}\tilde{F}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h} - C_F\tilde{F}_i, \quad (17)$$

with boundary conditions written as

$$\frac{\tilde{F}_1 - \tilde{F}_0}{h} = 0, \quad (18)$$

$$\frac{\tilde{F}_N - \tilde{F}_{N-1}}{h} = -G\tilde{F}_N. \quad (19)$$

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

- Finally we look for a backstepping transformation of the discretized original system into the discretization of the target system.
- This coordinate transformation is sought in the form

$$\tilde{F}_i \quad = \quad \tilde{E}_i - \alpha_{i-1}(\tilde{E}_1, ..., \tilde{E}_{i-1}), \tag{20}$$

- Substracting (17) from (11) we obtain $\dot{\alpha}_{i-1} = \dot{\tilde{E}}_i - \dot{\tilde{F}}_i$.
- Expressing the obtained equation in terms of $\alpha_{k-1} = \tilde{E}_k - \tilde{F}_k$, $k = i-1, i, i+1$ we can obtain the expression for α_i as

$$\begin{aligned} \alpha_i \quad = \quad & \frac{1}{D_{i+\frac{1}{2}} + \frac{D_i}{i}} \left[\left(2D_i + \frac{D_i}{i} + C_F h^2 \right) \alpha_{i-1} - D_{i-\frac{1}{2}} \alpha_{i-2} \right. \\ & \left. - h^2 g_i - h^2 C_F \tilde{E}_i + h^2 \dot{\alpha}_{i-1} \right], \end{aligned} \tag{21}$$

starting with $\alpha_0 = 0$ and where

$$\dot{\alpha}_{i-1} \quad = \quad \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \tilde{E}_k} \dot{\tilde{E}}_k. \tag{22}$$

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

- At this point we note the importance of the discretization method used to express $D_{i+\frac{1}{2}}$.

- The avoidance of writing these terms as functions of the state variables at point $i+1$ is fundamental to achieve the desired backstepping transformation (20).

- However, it is important to emphasize that although the usage of this specific discretization method is a requirement for the backstepping procedure, it does not represent any limitation at all.

Backstepping Control of Parabolic PDEs: Discretize $\rightarrow$ Design

- Similarly, substracting (19) from (13) and expressing the obtained equation in terms of $\alpha_{k-1} = \tilde{E}_k - \tilde{F}_k$, $k = i-1, i$ we can define the control $\Delta\tilde{E}_r$ as

$$\Delta\tilde{E}_r = \frac{\alpha_{N-1} - \alpha_{N-2}}{h} - k_E\tilde{E}_N - G\left(\tilde{E}_N - \alpha_{N-1}\right). \qquad (23)$$

- This expression for $\Delta\tilde{E}_r$ allows us to finally write the stabilizing laws for the boundary actuation

$$\tilde{E}_N = \alpha_{N-1} + \frac{1}{(1+Gh)}\left[\tilde{E}_{N-1} - \alpha_{N-2}\right]. \qquad (24)$$

- To show stability of the target system (14), we take the Lyapunov function candidate

$$V = \frac{1}{2}\int_0^a r\tilde{F}^2 dr$$

- Then we have

$$\begin{aligned}
\dot{V} &= \int_0^a r\tilde{F}\dot{\tilde{F}} dr, \\
&= \int_0^a r\tilde{F}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C_F\tilde{F}\right\} dr, \\
&= \tilde{F}rD\frac{\partial \tilde{F}}{\partial r}\Big|_0^a - \int_0^a rD\left(\frac{\partial \tilde{F}}{\partial r}\right)^2 dr - C_F\int_0^a r\tilde{F}^2 dr \\
&= aD(a)\tilde{F}(a)\tilde{F}_r(a) - \int_0^a rC_F\tilde{F}^2 dr - \int_0^a rD\tilde{F}_r^2 dr.
\end{aligned}$$

Backstepping Control of Parabolic PDEs: Discretize → Design

- Taking into account the boundary condition (16), we can write

$$\dot{V} \;=\; -C_F \int_0^a r\tilde{F}^2 dr - GaD(a)\tilde{F}^2(a) - \int_0^a rD\tilde{F}_r^2 dr \leq -C\frac{1}{2}\int_0^a r\tilde{F}^2 dr,$$

  and conclude that

$$\dot{V} \;\leq\; -CV$$

  showing that the system is asymptotically stable.

- The proof that the discretized target system (17) with boundary conditions (19) is asymptotically stable in l^2 norm would be completely analogous. The discrete Lyapunov function $V_d = \frac{1}{2}\sum_{i=0}^{N}\tilde{F}_i^2$ would be considered instead and following identical procedure the condition $\dot{V}_d \leq -CV_d$ would be obtained.

Backstepping Control of Parabolic PDEs: Discretize → Design

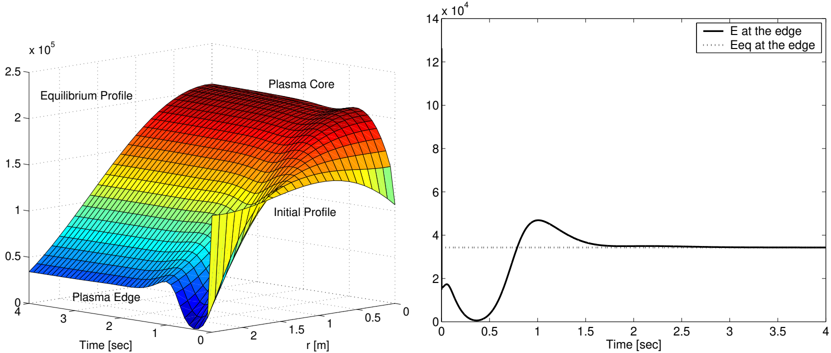

Figure 2: Simulation Results