Control of PDE Systems

Lecture 4 (Meetings 7 & 8)

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Linear PDEs

Two Dimensional Second Order Linear Elliptic PDE:

$$Lu = -\left[\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right)\right] + qu = f(x,y), \quad (x,y) \in \Omega \tag{1}$$

Boundary Conditions:

- Dirichlet

$$u(x,y) = \alpha(x,y), \quad \forall (x,y) \in \Gamma = \partial\Omega \tag{2}$$

- Neumann

$$\frac{\partial u}{\partial n}(x,y) = \beta(x,y), \quad \forall (x,y) \in \Gamma = \partial\Omega \tag{3}$$

- Robin

$$\frac{\partial u}{\partial n}(x,y) + \delta u(x,y) = \gamma(x,y), \quad \forall (x,y) \in \Gamma = \partial\Omega \tag{4}$$

Linear PDEs

One Dimensional Linear Parabolic PDE:

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + d(x,t)u = f(x,t), \quad (x,t) \in G \tag{5}$$

Initial and Boundary Conditions:

- Cauchy problem on the infinite domain

$$u|_{t=0} = \varphi(x), \quad -\infty < x < \infty \tag{6}$$

- Cauchy problem on the semi-infinite domain

$$u|_{t=0} = \varphi(x),\ 0 < x < \infty,\ \left[\alpha_1(t)\frac{\partial u}{\partial x} + \alpha_0(t)u\right]_{x=0} = \alpha_2(t) \tag{7}$$

- Cauchy problem on the finite domain

$$u|_{t=0} = \varphi(x), \quad 0 < x < l, \tag{8}$$

$$\left[\alpha_1(t)\frac{\partial u}{\partial x} + \alpha_0(t)u\right]_{x=0} = \alpha_2(t),\ \left[\beta_1(t)\frac{\partial u}{\partial x} + \beta_0(t)u\right]_{x=l} = \beta_2(t) \tag{9}$$

Linear PDEs

Hyperbolic PDE:

- First Order

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a \text{ is a constant} \tag{10}$$

- Second Order (Wave Equation)

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = 0, \quad a \text{ is a constant} \tag{11}$$

We note that

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial}{\partial t} + i\sqrt{a}\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} - i\sqrt{a}\frac{\partial}{\partial x}\right]u = 0, \tag{12}$$

where $i = \sqrt{-1}$.

Numerical Techniques

In solving PDEs, the primary challenge is to create effective and numerically stable approximations.

- **Finite Difference Method** uses finite difference equations to approximate derivatives on a prescribed grid.

- **Finite Element Method** uses the weak solution or the variational formation of PDEs to create finite dimensional approximations.

- A brief introduction of many numerical methods can be found in the paper by Eitan Tadmor:

  Numerical Methods for Partial Differential Equations.

Finite Difference Method

- The first order derivative is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \tag{13}$$

then a reasonable approximation could be

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}, \quad h \text{ is small.} \tag{14}$$

- It can be also derived from a Taylor series expansion around x_0:

$$f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f^{(2)}(x_0)}{2!}h^2 + \ldots \tag{15}$$

Then,

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{f^{(2)}(x_0)}{2!}h + \ldots \tag{16}$$

Approximation of order $\mathcal{O}(h)$. Can we obtain higher-order approximations?

Finite Difference Method

- Let us now compute two Taylor series expansions around x_0:

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f^{(2)}(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \dots \tag{17}$$

$$f(x_0-h) = f(x_0) - \frac{f'(x_0)}{1!}h + \frac{f^{(2)}(x_0)}{2!}h^2 - \frac{f^{(3)}(x_0)}{3!}h^3 + \dots \tag{18}$$

- By subtracting one from the other, we obtain

$$f(x_0+h) - f(x_0-h) = 2\frac{f'(x_0)}{1!}h + 2\frac{f^{(3)}(x_0)}{3!}h^3 + \dots \tag{19}$$

Then,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{f^{(3)}(x_0)}{3!}h^2 + \dots \tag{20}$$

We say that this is an approximation of order $\mathcal{O}(h^2)$.

Finite Difference Method - ODE Example

Given the ODE (Ordinary Differential Equations)

$$u'(x) = 3u(x) + 2, \quad 0 \leq x \leq l \tag{21}$$

we approximate the first order derivative by

$$u'(x) \approx \frac{u(x+h) - u(x)}{h}, \tag{22}$$

then we obtain the algebraic equation

$$u(x+h) = u(x) + h\left[3u(x) + 2\right]. \tag{23}$$

Given, for instance, $u(0)$, we can solve iteratively for $u(ih)$ for $i = 1, \ldots, N$ with $h = l/N$. Defining $u_i \triangleq u(ih)$, we can write

$$u_{i+1} = u_i + h\left[3u_i + 2\right], \tag{24}$$

which can be solved in a loop for $i = 0, \ldots, N-1$.

Numerical Techniques

Finite Difference Method - PDE Example (Heat Equation)

- Let us consider the normalized heat equation in one dimension,

$$\begin{array}{l} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) = u(1,t) = 0(\text{ boundary condition}), \\ u(x,0) = u_0(x)(\text{ initial condition}). \end{array} \tag{25}$$

- Let us also consider the grid partition: $u(x_j, t_n) = u_j^n$. Therefore,

$$x_0, x_1, \ldots, x_j, \ldots, x_J, \quad x_{j+1} - x_j \triangleq h, \tag{26}$$

$$t_0, t_1, \ldots, t_n, \ldots, t_N, \quad t_{n+1} - t_n \triangleq k. \tag{27}$$

Finite Difference Method - PDE Example (Heat Equation)

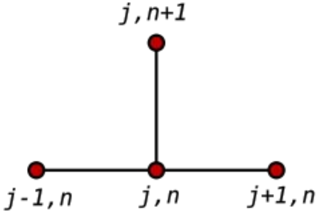
- Explicit Method

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \quad (28)$$

$$\Rightarrow u_j^{n+1} = (1-2r)u_j^n + ru_{j-1}^n + ru_{j+1}^n, \quad (29)$$

where $r = k/h^2$.

- This scheme is numerically stable only when $r \leq 1/2$. The numerical errors is of the order $\mathcal{O}(k) + \mathcal{O}(h^2)$.



Finite Difference Method - PDE Example (Heat Equation)

- Stability Analysis - Projection Matrix

$$u_j^{n+1} = ru_{j-1}^n + (1-2r)u_j^n + ru_{j+1}^n, \tag{30}$$

can be written as

$$U^{n+1} = AU^n \tag{31}$$

$$A = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & 1-2r \end{bmatrix}, \qquad U \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_J \end{bmatrix}. \tag{32}$$

The eigenvalues of A are given by

$$\lambda_j = 1 - 4r\sin^2\left(\frac{j\pi}{2J}\right), \qquad \mathrm{j} = 1, \dots, J. \tag{33}$$

We achieve $\max_j |\lambda_j| < 1$ only if $r < \frac{1}{2}$.

Finite Difference Method - PDE Example (Heat Equation)

- Stability Analysis - Von Neumann

$$u_j^{n+1} = ru_{j-1}^n + (1-2r)u_j^n + ru_{j+1}^n. \quad (34)$$

Let us assume $u_j^n = A^n e^{(i2\pi(jh))}$. Then,

$$\begin{aligned} A^{n+1}e^{i2\pi(jh)} &= rA^n e^{i2\pi((j-1)h)} + (1-2r)A^n e^{i2\pi(jh)} + rA^n e^{i2\pi((j+1)h)} \\ &= rA^n e^{i2\pi(jh)} e^{i2\pi(-h)} + (1-2r)A^n e^{i2\pi(jh)} + rA^n e^{i2\pi(jh)} e^{i2\pi(h)} \end{aligned}$$

and

$$\frac{A^{n+1}}{A^n} = re^{-i2\pi h} + 1 - 2r + re^{-i2\pi h} = 1 + r(e^{-i2\pi h} + e^{-i2\pi h} - 2) \quad (35)$$

Noting that

$$\sin\left(\frac{\theta}{2}\right) = \frac{e^{i\theta/2} - e^{-i\theta/2}}{2i} \rightarrow \sin^2\left(\frac{\theta}{2}\right) = -\frac{e^{i\theta} - e^{-i\theta} - 2}{4} \quad (36)$$

Finite Difference Method - PDE Example (Heat Equation)

We take $\theta \triangleq 2\pi h$ to write

$$G \triangleq \frac{A^{n+1}}{A^n} = 1 - 4r\sin^2\left(\frac{\theta}{2}\right) \tag{37}$$

Stability is achieved if $|G| < 1$, which implies

$$\left|1 - 4r\sin^2\left(\frac{\theta}{2}\right)\right| < 1 \tag{38}$$

Since $4r\sin^2\left(\frac{\theta}{2}\right)$ is always positive, this requires

$$4r\sin^2\left(\frac{\theta}{2}\right) < 2 \iff r < \frac{1}{2} \tag{39}$$

Finite Difference Method - PDE Example (Heat Equation)

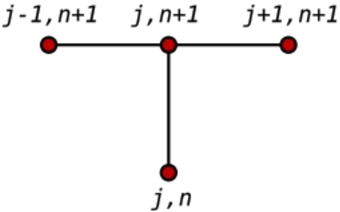
- Implicit Method

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} \tag{40}$$

$$\Rightarrow u_j^n = (1+2r)u_j^{n+1} - ru_{j-1}^{n+1} - ru_{j+1}^{n+1}, \tag{41}$$

where $r = k/h^2$.

- This scheme is always numerically stable. The numerical errors is of the order $\mathcal{O}(k) + \mathcal{O}(h^2)$.



Finite Difference Method - PDE Example (Heat Equation)

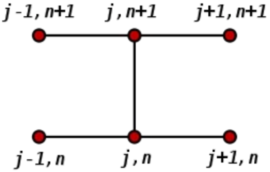
- Crank-Nicolson Method

$$\frac{u_j^{n+1}-u_j^n}{k}=\frac{1}{2}\left[\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}+\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{h^2}\right] \quad (42)$$

$$\Rightarrow (2+2r)u_j^{n+1}-ru_{j-1}^{n+1}-ru_{j+1}^{n+1}=(2-2r)u_j^n+ru_{j-1}^n+u_{j-1}^n+ru_{j+1}^n,$$

where $r=k/h^2$.

- This scheme is always numerically stable. The numerical errors is of the order $\mathcal{O}(k^2)+\mathcal{O}(h^4)$.



Finite Element Method - ODE Example

- Consider the following stationary equation

$$-u''(x) + u(x) = f(x), \quad x \in (0, \pi/2),\ u(0) = u'(\pi/2) = 0. \tag{43}$$

- Given any function $v \in V = \{v \in H^1((0, \pi/2)) | v(0) = 0\}$, we can define the weak formulation as

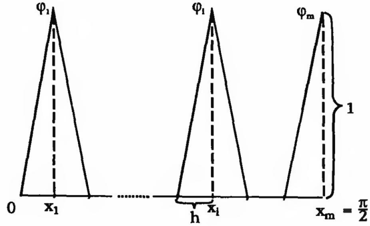
$$\int_0^{\pi/2} [-u''(x) + u(x)]\, v(x) dx = \int_0^{\pi/2} f(x) v(x) dx \tag{44}$$

$$\Rightarrow \int_0^{\pi/2} (u'(x)v'(x) + u(x)v(x))\, dx = \int_0^{\pi/2} f(x) v(x) dx, \tag{45}$$

for $u, v \in V$.

Finite Element Method - ODE Example

- We discretize the infinite dimensional space V by means of a piecewise linear functional space V_h.
- Linear Element ($h = \frac{\pi}{2m}$)



$$\begin{aligned} \phi_i(x) &= \frac{x}{h} - i + 1, \quad x \in [(i-1)h, ih], \\ \phi_i(x) &= -\frac{x}{h} + i + 1, \quad x \in [ih, (i+1)h], \\ \phi_i(x) &= 0, \quad \text{otherwise}. \end{aligned} \tag{46}$$

Finite Element Method - ODE Example

- We look for an approximate solution

$$u_h = \sum_{j=1}^{m} U_j \phi_j. \tag{47}$$

- Matrix form is obtained by replacing $u = u_h$ and $v \triangleq \phi_i$ in weak formulation:

$$\int_0^{\pi/2} \left(\sum_{j=1}^{m} U_j \phi_j'(x) \phi_i'(x) + \sum_{j=1}^{m} U_j \phi_j(x) \phi_i(x) \right) dx =$$

$$\int_0^{\pi/2} f(x)\phi_i(x)dx, \quad \phi_i, \phi_j \in V_h. \tag{48}$$

- We can obtain a linear system of equations $AU = (K+M)U = F$, where $U = [U_1\, U_2\, \ldots\, U_m]^T$ with $K, M \in \mathcal{R}^{m\times m}$ and $F \in \mathcal{R}^m$

$$K_{ij} \triangleq \int_0^{\pi/2} \phi_j'(x)\phi_i'(x)dx, M_{ij} \triangleq \int_0^{\pi/2} \phi_j(x)\phi_i(x)dx, F_i \triangleq \int_0^{\pi/2} f(x)\phi_i(x)dx.$$

Finite Element Method - PDE Example (Heat Equation)

- Given the normalized heat equation in one dimension,

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\\ &u(0,t) = u(1,t) = 0(\text{ boundary condition}),\\ &u(x,0) = u_0(x)(\text{ initial condition}).\end{aligned} \tag{49}$$

- The weak form is given by

$$\left(\frac{\partial u}{\partial t}, v\right)_{L^2(0,1)} + a(u,v) = 0, \quad a(u,v) = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx, \forall v \in V. \tag{50}$$

- We look for an approximate solution

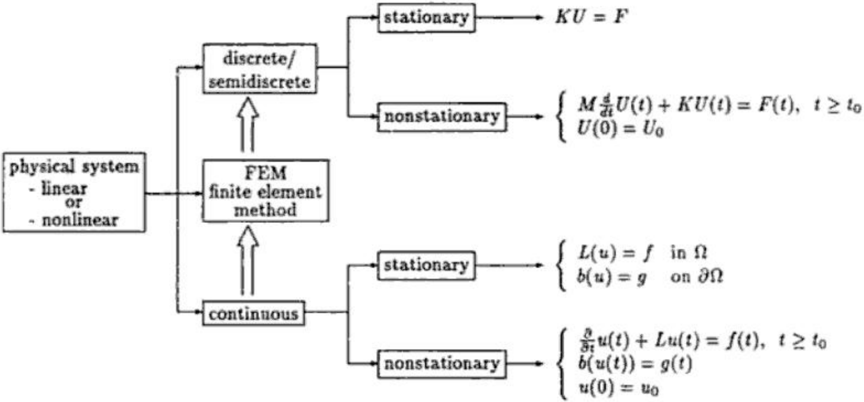
$$u \approx u_h = \sum_{i=1}^{m} u_i(t)\phi_i(x). \tag{51}$$

By choosing $v = \phi_i$, we obtain the linear system $M\frac{dU}{dt} + KU = 0$, where

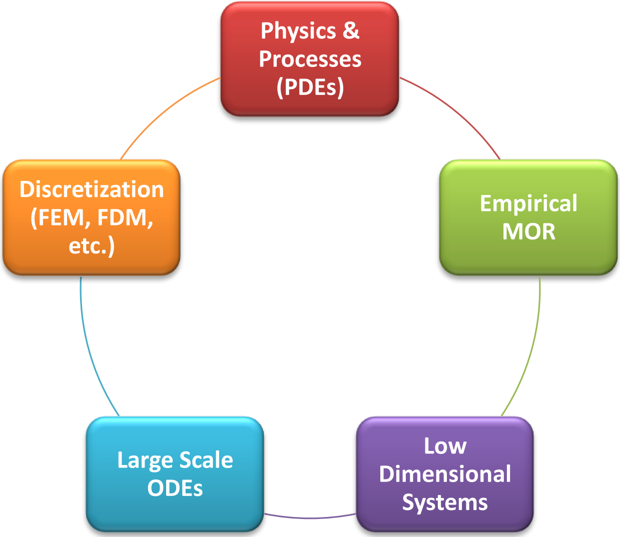
$$U = (u_i)_{i=1}^m, \quad M_{ij} = \int_0^1 \phi_i(x)\phi_j(x)dx, \quad K_{ij} = \int_0^1 \phi_i'(x)\phi_j'(x)dx$$

Note: t has NOT been discretized! OK for control design. Not for simulation.

Finite Element Method



Control/Optimization Based on ROM



Control/Optimization Based on ROM

Two different approaches:

- Discretization-then-Design
- Design-then-Discretization

Reference:

Discrete Concepts in PDE Constrained Optimization,
Chapter 3 in the book by M. Hinze et al.,
Optimization with PDE Constraints.