Control of PDE Systems

Lecture 3 (Meetings 5 & 6)

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Control Lyapunov Function (CLF)

Let us consider the following system,

$$\dot{\eta} = f(\eta, u), \quad \eta \in R^n, \quad u \in R, \quad f(0,0) = 0, \tag{1}$$

the following two tasks are equivalent:

Task 1: Find a feedback control law $u = \phi(\eta)$ **such that the equilibrium** $\eta = 0$ **of the closed-loop system**

$$\dot{\eta} = f(\eta, \phi(\eta)) \tag{2}$$

is globally asymptotically stable.

Task2 : Find a feedback control law $u = \phi(\eta)$ **and a Lyapunov function candidate** $V(\eta)$ **such that**

$$\dot{V} = \frac{\partial V}{\partial \eta}(\eta) f(\eta, \phi(\eta)) \leq -W(\eta), \quad W(\eta) \textbf{ positive definite} \tag{3}$$

with $W(\eta)$ **positive definite.**

A system for which good choices of $V(\eta)$, $W(\eta)$ exist is said to possess a CLF.

Backstepping

Assumption: There exist a stabilizing state feedback control law $\phi(\eta)$, with $\phi(0) = 0$, and a Lyapunov function $V(\eta)$ s.t.

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad W(\eta) \text{ positive definite} \tag{4}$$

Lemma [K] 14.2: Integrator Backstepping

$$\dot{\eta} = f(\eta) + g(\eta)\xi \tag{5}$$

$$\dot{\xi} = u \tag{6}$$

There is a whole integrator between u and ξ. Under the previous assumption, the system has a CLF

$$V_a(\eta, \xi) = V(\eta) + \frac{1}{2}(\xi - \phi(\eta))^2, \qquad \text{(a: augmented)} \tag{7}$$

and the corresponding feedback that gives global asymptotical stability is

$$u = -c(\xi - \phi(\eta)) + \frac{\partial \phi}{\partial \eta}(\eta)[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta}g(\eta), \qquad c > 0 \tag{8}$$

Moreover, if all the assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be globally asymptotically stable.

Backstepping

Proof: From the book.

Backstepping

Backstepping: We have a "virtual" control ξ and we have to go back through an integrator.

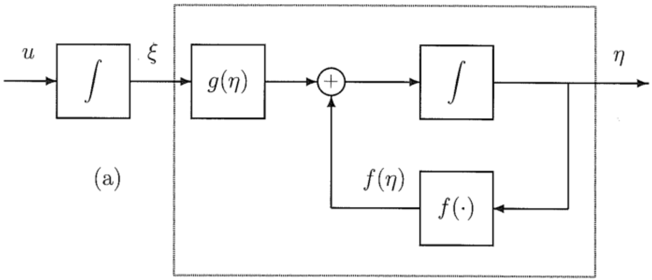

Figure 1: Original block diagram.

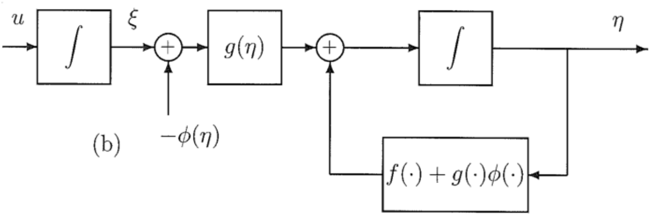

Figure 2: Block diagram after introducing $\phi(\eta)$.

Backstepping

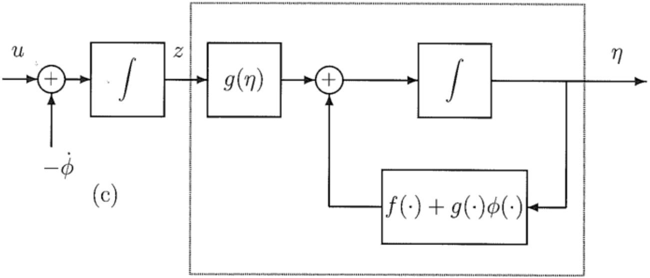

Figure 3: Block diagram after "backstepping" $-\phi(\eta)$ through integrator.

Backstepping

Example:

$$\dot{x}_1 = x_1 x_2 \quad (9)$$

$$\dot{x}_2 = u \quad (10)$$

Let us take x_2 as the virtual controller for the x_1 system, i.e.

$$\dot{x}_1 = x_1 v \quad (11)$$

What v control law could we take to stabilize the equilibrium $x_1 = 0$?

Could we take $v = \phi(x_1) = -1$? Yes, but $\phi(x_1) \neq 0$ when $x_1 = 0$.

Let us take

$$v = \phi(x_1) = -x_1^2 \quad (12)$$

Then,

$$\dot{x}_1 = -x_1^3 \quad (13)$$

Take $V(x_1) = \frac{1}{2}x_1^2 > 0$. Then, $\dot{V}(x_1) = -x_1^4$.

Let us define $z \triangleq x_2 - \phi(x_1) \Rightarrow x_2 = z + \phi(x_1) = z - x_1^2$. Then,

$$\dot{z} = \dot{x}_2 - \frac{\partial \phi(x_1)}{\partial x_1}\dot{x}_1 = u + 2x_1(x_1 x_2) = u + 2x_1^2(z - x_1^2) \tag{14}$$

and we can write

$$\begin{array}{rcl} \dot{x}_1 & = & x_1 x_2 \\ \dot{x}_2 & = & u \end{array} \Rightarrow \begin{array}{rcl} \dot{x}_1 & = & -x_1^3 + x_1 z \\ \dot{z} & = & u + 2x_1^2(z - x_1^2) \end{array} \tag{15}$$

Take $V_a(x_1, z) = V(x_1) + \frac{1}{2}z^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z^2 > 0$. Then,

$$\begin{aligned} \dot{V}_a(x_1, z) = x_1\dot{x}_1 + z\dot{z} &= x_1(-x_1^3 + x_1 z) + z(u + 2x_1^2(z - x_1^2)) && (16) \\ &= -x_1^4 + z(u + x_1^2 + 2x_1^2(z - x_1^2)) && (17) \end{aligned}$$

What v control law could we take to make $\dot{V}_a(x_1, z) < 0$? Let us make

$$u + x_1^2 + 2x_1^2(z - x_1^2) = -z \iff u = -z - x_1^2 - 2x_1^2(z - x_1^2) = -z - x_1^2 - 2x_1^2 x_2 \tag{18}$$

Then, $\dot{V}_a(x_1, z) = -x_1^4 - z^2 < 0 \Rightarrow (x_1, z) \to 0 \Rightarrow (x_1, x_2) \to 0$.

Example [K] 14.8:

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 \quad (19)$$

$$\dot{x}_2 = u \quad (20)$$

Backstepping

In the case of more than one integrator

$$\dot{\eta} = f(\eta) + g(\eta)\xi_1 \quad (21)$$
$$\dot{\xi}_1 = \xi_2 \quad (22)$$
$$\vdots \quad (23)$$
$$\dot{\xi}_{n-1} = \xi_n \quad (24)$$
$$\dot{\xi}_n = u \quad (25)$$

we only have to apply the backstepping lemma n times.

Backstepping

Example [K] 14.9:

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 \quad (26)$$

$$\dot{x}_2 = x_3 \quad (27)$$

$$\dot{x}_3 = u \quad (28)$$

Backstepping

In the more general case

$$\dot{x} = f(x) + g(x)\xi \quad (29)$$
$$\dot{\xi} = f_a(x,\xi) + g_a(x,\xi)u \quad (30)$$

If $g_a(x,\xi) \neq 0$ over the domain of interest, the input transformation

$$u = \frac{1}{g_a(x,\xi)}[v - f_a(x,\xi)] \quad (31)$$

will reduce the system to

$$\dot{x} = f(x) + g(x)\xi \quad (32)$$
$$\dot{\xi} = v \quad (33)$$

and the backstepping lemma can be applied.

Backstepping

Strict Feedback Systems:

$$\dot{x}_i = x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n-1 \tag{34}$$
$$\dot{x}_n = u + \phi_n(x) \tag{35}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T$, $\phi_i(\bar{x}_i)$ are smooth and $\phi_i(0) = 0$.
We have a local triangular structure:

$$\dot{x}_1 = x_2 + \phi_1(x_1) \tag{36}$$
$$\dot{x}_2 = x_3 + \phi_2(x_1, x_2) \tag{37}$$
$$\vdots \tag{38}$$
$$\dot{x}_n = u + \phi_n(x_1, x_2, \ldots, x_n) \tag{39}$$

Linear part: Brunovsky canonical form $\Rightarrow$ feedback linearizable

The control law

$$z_i = x_i - \alpha_{i-1}(\bar{x}_{i-1}) \quad \alpha_0 = 0 \tag{40}$$

$$\alpha_i(\bar{x}_i) = -z_{i-1} - c_i z_i - \phi_i + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j), \quad c_i > 0 \tag{41}$$

$$u = \alpha_n \tag{42}$$

guarantees global asymptotic stability of $x = 0$.

Proof:

$$\dot{z}_i = \dot{x}_i - \dot{\alpha}_{i-1}(\bar{x}_{i-1}) \tag{43}$$

$$= x_{i+1} + \phi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j(\bar{x}_j)) \tag{44}$$

$$= z_{i+1} + \alpha_i(\bar{x}_i) + \phi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j(\bar{x}_j)) \tag{45}$$

Backstepping

By picking

$$\alpha_i(\bar{x}_i) = -z_{i-1} - c_i z_i - \phi_i(\bar{x}_i) + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j(\bar{x}_j)), \quad c_i > 0 \tag{46}$$

we obtain

$$\dot{z}_i = -z_{i-1} - c_i z_i + z_{i+1} \tag{47}$$

We only need to define

$$z_0 = 0, \quad z_{n+1} = u - \alpha_n = 0 \tag{48}$$

to be able to write

$$\dot{z} = \begin{bmatrix} -c_1 & 1 & & & \\ -1 & -c_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & -c_n \end{bmatrix} z \triangleq Az \tag{49}$$

This is a linear matrix with special structure!!!

Backstepping

Take

$$V = \frac{1}{2}\sum_{i=1}^{n} z_i^2 \tag{50}$$

and compute

$$\dot{V} = \sum_{i=1}^{n} z_i \dot{z}_i = -\sum_{i=1}^{n} c_i z_i^2 - \sum_{i=1}^{n} z_i z_{i-1} + \sum_{i=1}^{n} z_i z_{i+1} = -\sum_{i=1}^{n} c_i z_i^2 < 0 \tag{51}$$

Since

$$z_i = x_i - \alpha_{i-1}(x_1, x_2, \ldots, x_{i-1}) \tag{52}$$

we note that

$$\frac{\partial z}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ X & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ldots & X & 1 & 0 \\ X & \ldots & \ldots & X & 1 \end{bmatrix} \tag{53}$$

Then, $z(x)$ is smoothly invertible and we conclude that $x = 0$ is g.a.s. Note that we can find another transformation $A \xrightarrow{T} A^*$ with desired A^*.

Backstepping

The technique can be extended to more general **Strict Feedback Systems:**

$$\dot{x}_i = \psi_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n-1 \tag{54}$$

$$\dot{x}_n = \psi_n(x)u + \phi_n(x) \tag{55}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T$ ($\bar{x}_n = x$), $\phi_i(\bar{x}_i)$ are smooth and $\phi_i(0) = 0$, and $\psi_i(\bar{x}_i) \neq 0$ for $i = 1, \ldots, n$ over the domain of interest.

Note: More in the book.