Control of PDE Systems

Lecture 2 (Meetings 3 & 4)

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Approaches to Control of Distributed Parameter Systems

- Controllability
- Optimal control
- Abstract approaches based on semigroups
- Frequency-domain approaches based on robust control (not natural because PDEs come in time domain and conversion to frequency domain is hard; model reduction, implied by the robust control approach, is also hard)
- "Boundary damper" controllers (for a limited class of systems and under a very limited actuation architecture)
- Very few of these results have ever been tested in simulations.

Some Well Known Books

- Krstic and Smyshlyaev;
- Christofides;
- Lions;
- Komornik;
- Curtain and Zwart;
- Lasiecka and Triggiani;
- Bensoussan, Da Prato, Delfour, and Mitter;
- Li and Yong;
- van Keulen;
- Luo, Guo, and Morgul;
- Lagnese;
- Lasiecka;
- Banks, Smith, and Wang;
- de Queiroz, Dawson, Nagarkatti, and Zhang;
- Aamo and Krstic;
- Gunzburger;

Applications

- Flexible structures (aerospace, civil, AFM)
- Chemical process control
- Fluids, aerodynamics, turbulence, propulsion, acoustics
- Quantum control
- Delays (machine tool chatter, combustion instabilities, etc.)

Classes of PDEs

- Parabolic (heat transfer, chemical reactions, etc)
- Hyperbolic (waves—acoustics, strings, etc)
- Other "odd" equations (most physically relevant problems are):
  - Navier-Stokes
  - Korteweg-de Vries
  - Kuramoto-Sivashinsky
  - Some beam/plate/shell models

Actuator Location

- Boundary control
- In-domain control (a few actuators)
- Distributed control (lots of actuators)
- Diffusivity control

Stability of PDE Systems

- No useful "general Lyapunov theory" for infinite dimensional systems
- Spatial norms
- Inequalities
  - Poincare
  - Agmon
  - Sobolev
- Energy boundedness vs. pointwise (in space) boundedness

Choice of Boundary Control

- Dirichlet (common in fluid problems)

- Neumann (common in thermal problems)

Static vs. Dynamic Behavior in PDEs

- Equilibrium/static problems = PDEs themselves (or, in the 1D case, ODEs).

Nonlinear Issues

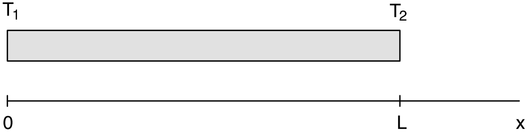
- Blow up in time (superlinear nonlinearities like in chemical reactions)

- Blow up in space (shock waves—Burgers, etc.)

- Boundedness despite quadratic nonlinearities (Navier-Stokes)

Introduction

Simplest physical model: heating rod



Heat equation	$T_\tau(\xi,\tau) = \epsilon T_{\xi\xi}(\xi,\tau)$	(1)
Left boundary condition	$T(0,\tau) = T_1$	(2)
Right boundary condition	$T(L,\tau) = T_2$	(3)
Initial condition	$T(\xi,0) = T_0(\xi)$	(4)

We want to represent this equation in a form suitable for our course:

- Nondimensional variables that describe the error between the unsteady temperature and the equilibrium profile of the temperature.

Introduction

Procedure:

1. Scale ξ to normalize length: $x = \frac{\xi}{L} \Rightarrow T_\tau = \frac{\epsilon}{L^2} T_{xx}$
2. Scale τ to normalize diffusion coefficient: $t = \frac{\epsilon}{L^2}\tau \Rightarrow T_t = T_{xx}$
3. Find steady-state solution $\bar{T}$

$$\begin{array}{rcl} \bar{T}''(x) & = & 0 \\ \bar{T}(0) & = & T_1 \\ \bar{T}(1) & = & T_2 \end{array} \Rightarrow \bar{T} = T_1 + x(T_2 - T_1) \tag{5}$$

4. Introduce the error variable $w = T - \bar{T}$

$$w_t = w_{xx} \tag{6}$$
$$w(0) = 0 \tag{7}$$
$$w(1) = 0 \tag{8}$$

   with initial condition $w(x, 0) = w_0$.

5. Finally, suppress time and space dependence where it does not lead to confusion; i.e., by w, $w(0)$ we always mean $w(x, t)$, $w(0, t)$, respectively, unless specifically stated otherwise.

Basic Types of Boundary Conditions

Basic types of boundary conditions for PDEs in one dimension:

- Dirichlet: $w(0) = 0$ (temperature)
- Neumann: $w_x(0) = 0$ (heat flux)
- Robin(mixed): $w(0) + qw_x(0) = 0$

The control design approach will depend on the type of boundary condition.

Stability of PDEs

Heat equation

$$w_t = w_{xx} \tag{9}$$

$$w(0) = 0 \tag{10}$$

$$w(1) = 0 \tag{11}$$

As in finite dimension, there are two ways to analyze stability properties:

- Find the exact solution
  - Usually not possible
- Use Lyapunov theory to show stability without solving the PDE
  - There is no general Lyapunov theory for PDEs
- For this simple plant both methods can be applied
  - Not so for more complicated systems

Lyapunov Stability

Most common Lyapunov function for PDEs is L_2 spatial norm:

$$V = \frac{1}{2}\int_0^1 w^2(x)dx \triangleq \frac{1}{2}\|w\|^2 \tag{12}$$

Time derivative along the solutions:

$$\begin{aligned}
\dot{V} = \frac{dV}{dt} &= \int_0^1 w(x)w_t(x)dx && (13)\\
&= \int_0^1 w(x)w_{xx}(x)dx \quad (w_t = w_{xx}) && (14)\\
&= w(x)w_x(x)|_0^1 - \int_0^1 w_x^2(x)dx \quad \text{(Integration by Parts)} && (15)\\
&= -\int_0^1 w_x^2(x)dx \quad (w(0) = w(1) = 0) && (16)
\end{aligned}$$

So the system is stable, but is it asymptotically stable? Not demonstrated yet!

Need to express $\|w_x\|$ in terms of $\|w\|$.

Integration by parts

Product Rule:

$$\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} \iff u\frac{dv}{dx} = \frac{d(uv)}{dx} - \frac{du}{dx}v \tag{17}$$

Integrating both sides of the equation we obtain

$$\int_0^1 u\frac{dv}{dx}dx = \int_0^1 \frac{d(uv)}{dx}dx - \int_0^1 \frac{du}{dx}vdx \tag{18}$$

$$= uv|_0^1 - \int_0^1 \frac{du}{dx}vdx \tag{19}$$

Useful Inequalities

Young's Inequality (special case):

$$ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2 \tag{20}$$

Cauchy-Schwarz Inequality:

$$\int_0^1 uw dx \leq \left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 w^2 dx\right)^{\frac{1}{2}} = \|u\|\|w\| \tag{21}$$

And using Young's Inequality $\|u\|\|w\| \leq \frac{\gamma}{2}\|u\|^2 + \frac{1}{2\gamma}\|w\|^2$,

$$\int_0^1 uw dx \leq \left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 w^2 dx\right)^{\frac{1}{2}} = \|u\|\|w\| \leq \frac{\gamma}{2}\|u\|^2 + \frac{1}{2\gamma}\|w\|^2 \tag{22}$$

Useful Inequalities

Poincaré Inequality:

$$\int_0^1 w^2 dx \leq 2w^2(1) + 4\int_0^1 w_x^2 dx \qquad (23)$$

$$\int_0^1 w^2 dx \leq 2w^2(0) + 4\int_0^1 w_x^2 dx \qquad (24)$$

In particular, if one of the boundary conditions is zero, then

$$\|w\| \leq 2\|w_x\| \qquad (25)$$

Cauchy-Schwarz Inequality

Proof of Cauchy-Schwarz Inequality:

$$\forall y : \quad 0 \leq (yu(x) + w(x))^2 \Rightarrow \int_0^1 (yu(x) + w(x))^2 \, dx$$

Then,

$$\forall 0 \leq y^2 \int_0^1 u(x)^2 dx + 2y \int_0^1 u(x) w(x) dx + \int_0^1 w(x)^2 dx = Ay^2 + 2By + C$$

where

$$A = \int_0^1 u(x)^2 dx, \qquad B = \int_0^1 u(x) w(x) dx, \qquad C = \int_0^1 w(x)^2 dx$$

The quadratic function $f(y) = Ay^2 + 2By + C$ is non-negative for all y. The discriminant of $f(y) = 0$ is given by $(2B)^2 - 4AC$. It the discriminant were positive, there would be two distinct real roots, $\lambda_1 < \lambda_2$, and $f = A(y - \lambda_1)(y - \lambda_2)$ would be negative somewhere. When y is between λ_1 and λ_2, $(y - \lambda_1)$ is positive and $(y - \lambda_2)$ is negative, and their product is negative. When y is greater than both λ_1 and λ_2, $(y - \lambda_1)$ and $(y - \lambda_2)$ are positive, and their product is positive. This means that the sign of $f(y)$ must change at $y = \lambda_2$. But we have $f(y) \geq 0$!!! So the discriminant cannot be positive. Therefore,

$$(2B)^2 - 4AC \leq 0 \iff B^2 \leq AC$$

Cauchy-Schwarz Inequality

Then

$$\left(\int_0^1 u(x)w(x)dx\right)^2 \le \int_0^1 u(x)^2dx \int_0^1 w(x)^2dx$$
$$\int_0^1 u(x)w(x)dx \le \sqrt{\int_0^1 u(x)^2dx}\sqrt{\int_0^1 w(x)^2dx}$$

Another useful inequality:

$$\gamma a^2 - 2ab + \frac{1}{\gamma}b^2 = \left(\sqrt{\gamma}a - \sqrt{\frac{1}{\gamma}}b\right)^2 \ge 0$$

Therefore,

$$\gamma a^2 + \frac{1}{\gamma}b^2 \ge 2ab$$

or

$$ab \le \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2$$

Poincare Inequality

Proof of Poincare Inequality:

$$\begin{aligned}\int_0^1 w^2 dx &= xw^2|_0^1 - 2\int_0^1 xww_x\,dx \\ &= w^2(1) - 2\int_0^1 xww_x\,dx \\ &\leq w^2(1) + \frac{1}{2}\int_0^1 w^2\,dx + 2\int_0^1 x^2 w_x^2\,dx\end{aligned}$$

We get

$$\begin{aligned}\frac{1}{2}\int_0^1 w^2 dx &\leq w^2(1) + 2\int_0^1 x^2 w_x^2 dx \\ &\leq w^2(1) + 2\int_0^1 w_x^2 dx\end{aligned}$$

Finally

$$\int_0^1 w^2 dx \leq 2w^2(1) + 4\int_0^1 w_x^2\,dx$$

Lyapunov Stability

Back to Lyapunov function:

$$\begin{aligned}
\dot{V} &= -\int_0^1 w_x^2(x)dx && (26)\\
&\leq -\frac{1}{4}\int_0^1 w^2 dx \quad \text{Poincare Inequality } (w(0) = w(1) = 0) && (27)\\
&\leq -\frac{1}{2}V && (28)
\end{aligned}$$

Therefore

$$V(t) \leq V(0)e^{-t/2} \quad \text{or} \quad \|w(x,t)\| \leq \|w(x,0)\|e^{-t/4} \tag{29}$$

- We showed that $\|w\| \to 0$ as $t \to \infty$ (exponential stability in L_2).
- Figure 1 illustrates demonstrated stability results.

Typical Response

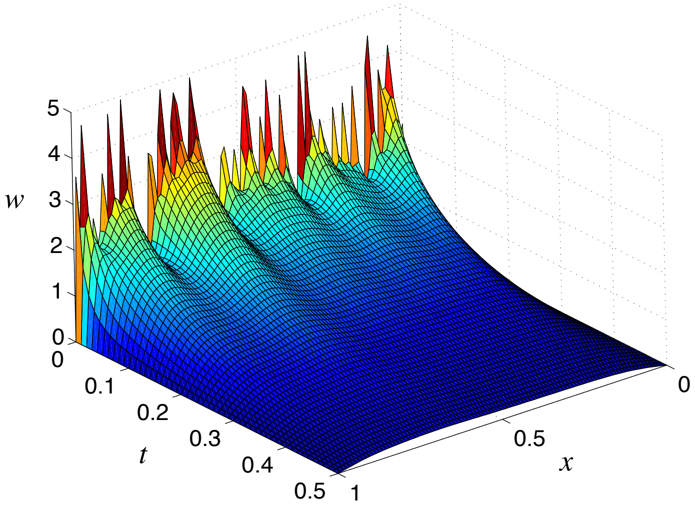


Figure 1: Response of a heat equation to a nonsmooth initial condition.

Lyapunov Stability

- By using Lyapunov stability theory we predicted the overall decay of the solution without knowledge of the exact solution $w(x,t)$ for a specific initial condition $w0(x)$.
- For PDEs, the L_2 form of stability in (29) is just one of the many possible (non-equivalent) forms of stability.
- Lyapunov function (12) is just one of the many possible choices, a well known feature of the Lyapunov method for ODEs.
- Nevertheless, the L_2 stability, quantified by (12)) and (29), is usually the easiest one to prove for a vast majority of PDEs, and an estimate of the form (29) is often needed before proceeding to study stability in higher norms.

Once again, we showed that $\|w\| \to 0$ as $t \to \infty$ (exponential stability in L_2).

This does not imply that $w(x,t) \to 0$ as $t \to \infty$ for all x. There could be "unbounded spikes" for some x along the spatial domain (on a set of measure zero) that do not contribute to the L_2-norm (unlikely to occur for the heat equation as shown in Figure 1).

Pointwise Stability

Would like to show that

$$\max_{x\in[0,1]} |w(x,t)| \leq K\mathrm{e}^{-\frac{t}{4}} \max_{x\in[0,1]} |w(x,0)|$$

This result cannot be proved. However, it is possible to show a slightly weaker result

$$\max_{x\in[0,1]} |w(x,t)| \leq K\mathrm{e}^{-\frac{t}{4}} \|w_0\|_{H_1}$$

We define H_1 norm as

$$\|w\|_{H_1} := \int_0^1 w^2\, dx + \int_0^1 w_x^2\, dx$$

Note that by using Poincare inequality it is possible to drop the integral of w^2 for most problems

Agmon Inequality

$$\max_{x\in[0,1]} |w(x)|^2 \leq w^2(0) + 2\|w\|\|w_x\|$$
$$\max_{x\in[0,1]} |w(x)|^2 \leq w^2(1) + 2\|w\|\|w_x\|$$

Proof:

$$\int_0^x w(\xi) w_\xi(\xi)\, d\xi = \frac{1}{2} w^2(\xi)|_0^x = \frac{1}{2} w^2(x) - \frac{1}{2} w^2(0)$$

Using triangle inequality we get

$$\frac{1}{2} w^2(x) \leq \frac{1}{2} w^2(0) + \int_0^x |w(\xi||w_\xi(\xi)|\, d\xi$$
$$w^2(x) \leq w^2(0) + 2\int_0^1 |w(\xi)||w_\xi(\xi)|\, dx$$
$$\max_{x\in[0,1]} |w(x)|^2 \leq w^2(0) + 2\|w\|\|w_x\|$$

Pointwise Stability

Back to our problem:

$$\begin{aligned} w_t &= w_{xx} \\ w(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Let us use the Lyapunov function

$$V = \frac{1}{2}\int_0^1 w^2(x)\,dx + \frac{1}{2}\int_0^1 w_x^2(x)\,dx$$

$$\begin{aligned} \dot{V} &= \int_0^1 w w_{xx}\,dx + \int_0^1 w_x w_{tx}\,dx \\ &= w(x)w_x(x)|_0^1 - \int_0^1 w_x^2\,dx + w_t(x)w_x(x)_0^1 - \int_0^1 w_{xx}w_t\,dx \\ &= -\int_0^1 w_x^2\,dx - \int_0^1 w_{xx}^2\,dx \\ &\leq -\frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|w_x\|^2 \\ &\leq -\frac{1}{8}\|w\|^2 - \frac{1}{2}\|w_x\|^2 \\ &\leq -\frac{1}{4}V \end{aligned}$$

Pointwise Stability

We have

$$\|w(t)\|^2 + \|w_x(t)\|^2 \leq \mathrm{e}^{-t/2} \left(\|w_0\|^2 + \|w_{0x}\|^2 \right)$$

where $w_0 = w(x,0)$ is the initial condition.

Finally,

$$\begin{aligned} \max_{x\in[0,1]} |w(x,t)|^2 &\leq 2\|w(t)\|\|w_x(t)\| \quad \text{(Agmon inequality)} \\ &\leq \|w(t)\|^2 + \|w_x(t)\|^2 \\ &\leq \mathrm{e}^{-t/2} \left(\|w_0\|^2 + \|w_{0x}\|^2 \right) \end{aligned}$$

We showed that the equilibrium $w \equiv 0$ is asymptotically stable for all $x \in [0,1]$.

Exact Solutions

Exist mostly for the plants with constant parameters.

Two standard methods for finding exact solutions: separation of variables and Laplace transform.

Separation of Variables

Heat equation with reaction:

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

Postulate the solution in the form $u(x,t) = X(x)T(t)$.

Exact Solutions

Substitute $u(x,t) = X(x)T(t)$ in the equation:

$$X(x)\dot{T}(t) = X''(x)T(t) + \lambda X(x)T(t)$$

Divide by $X(x)T(t)$:

$$\frac{\dot{T}}{T} = \frac{X'' + \lambda X}{X} = \sigma$$

ODE for T:

$$\begin{aligned} \dot{T} &= \sigma T \\ T &= e^{\sigma t} \quad \text{(without loss of generality)} \end{aligned}$$

ODE for X:

$$\begin{aligned} X'' + (\lambda - \sigma)X &= 0 \\ X(0) &= X(1) = 0 \end{aligned}$$

Solution for $X(x)$:

$$X(x) = A\sin(\sqrt{\lambda - \sigma}x) + B\cos(\sqrt{\lambda - \sigma}x)$$

Exact Solutions

$$X(x) = A\sin(\sqrt{\lambda-\sigma}x) + B\cos(\sqrt{\lambda-\sigma}x)$$

$$\begin{aligned}
X(0) = 0 \quad &\Rightarrow \quad B = 0 \\
X(1) = 0 \quad &\Rightarrow \quad A\sin(\sqrt{\lambda-\sigma}) = 0 \\
&\Rightarrow \quad \sqrt{\lambda-\sigma} \; = \; \pi n, \quad \text{where } n = 0, 1, 2, \ldots \\
&\Rightarrow \quad \sigma = \lambda - \pi^2 n^2
\end{aligned}$$

Solution

$$u_n(x,t) = A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x), \quad n = 0, 1, 2, \ldots$$

Since the PDE is linear, the sum of solutions is also a solution. Therefore the formal general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x)$$

To find A_n we use the knowledge of the initial condition $u(x,0) = u_0(x)$

Exact Solutions

Set $t = 0 \;\Rightarrow\; u_0(x) = \sum_{n=1}^{\infty} A_n \sin(\pi n x)$

Multiply both sides with $\sin(\pi m x) \;\Rightarrow\; u_0(x)\sin(\pi m x) = \sin(\pi m x)\sum_{n=1}^{\infty} A_n \sin(\pi n x)$

Use the orthogonality property $\int_0^1 \sin(\pi m x)\sin(\pi n x)dx = \left\{ \begin{array}{cc} 1/2 & n = m \\ 0 & n \neq m \end{array} \right\}$
to get

$$\int_0^1 u_0(x)\sin(\pi m x)dx = \frac{1}{2}A_m$$

The exact solution is

$$u(x,t) = 2\sum_{n=1}^{\infty} \mathrm{e}^{\overbrace{(\lambda - \pi^2 n^2)}^{\text{eigenvalues}}t} \underbrace{\sin(\pi n x)}_{\text{eigenfunctions}} \overbrace{\int_0^1 \sin(\pi n x) u_0(x)\, dx}^{\text{effect of initial conditions}}$$

The stability condition is $\lambda < \pi^2$. Note that it is much less conservative than the one obtained using Lyapunov method (which gives $\lambda < 1/4$).

Exact Solutions

Example. Find values of the parameter g for which the system

$$\begin{aligned} u_t &= u_{xx} + g u(0) \\ u_x(0) &= u(1) = 0 \end{aligned}$$

is unstable.

Let $u(x,t) = e^{\sigma t} X(x)$. Substitute this solution in the PDE to get an ODE

$$X''(x) - \sigma X = -g X(0)$$

which has a general solution

$$X(x) = A \sinh(\sqrt{\sigma} x) + B \cosh(\sqrt{\sigma} x) + \frac{g}{\sigma} X(0)$$

To find B, let $x = 0$:

$$X(0) = B + \frac{g}{\sigma} X(0) \quad \Rightarrow \quad B = X(0)\left(1 - \frac{g}{\sigma}\right)$$

Boundary condition at $x = 0$ gives

$$X'(0) = 0 \quad \Rightarrow \quad A = 0$$

Exact Solutions

We have

$$X(x) = X(0)\left[\frac{g}{\sigma} + \left(1 - \frac{g}{\sigma}\right)\cosh(\sqrt{\sigma}x)\right]$$

Boundary condition at $x = 1$ gives

$$X(1) = 0 \quad \Rightarrow \quad \cosh(\sqrt{\sigma}) \quad = \quad \frac{g}{g - \sigma}$$

This equation cannot be solved in closed form. But we can still find the region of stability.

Solve for g in terms of σ:

$$g \quad = \quad \frac{\sigma\cosh(\sqrt{\sigma})}{\cosh(\sqrt{\sigma}) - 1}$$

Take the limit $\sigma \to 0$:

$$g \quad = \quad \lim_{\sigma\to 0}\frac{\sigma\cosh(\sqrt{\sigma})}{\cosh(\sqrt{\sigma}) - 1} = \lim_{\sigma\to 0}\frac{\sigma(1+\sigma/2)}{1+\sigma/2-1} = 2$$

Therefore, the PDE is unstable for $g > 2$.